Problem Set 8: Complex Numbers

<u>Goal</u>: Become familiar with math operations using complex numbers; see how complex numbers can be used to show the frequency response of an RC circuit.

Note: This PSet will be much easier if you have already watched the lectures on complex numbers.

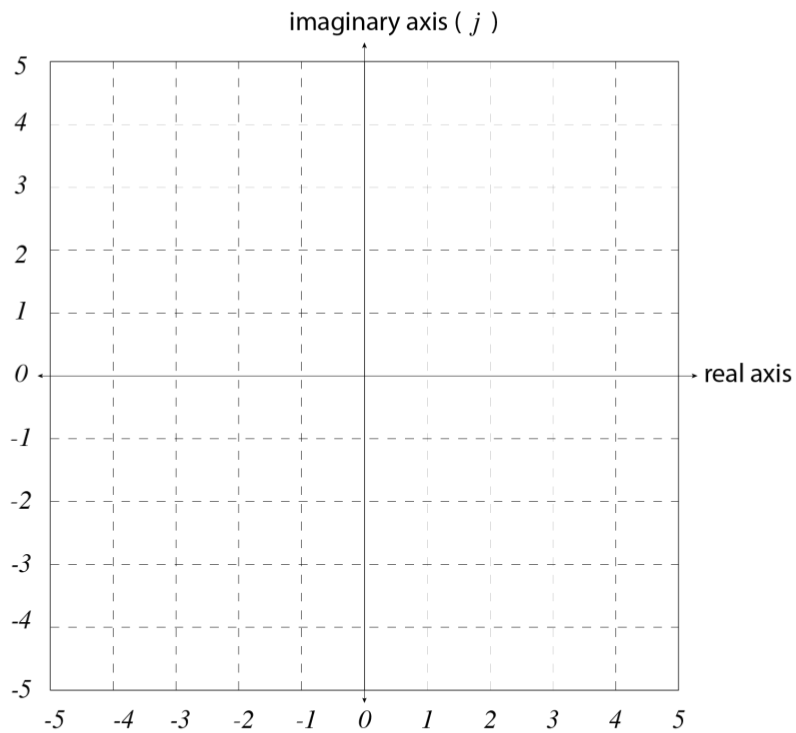
Deliverables: This worksheet and two plots.

Part I: Basic Operations with complex numbers

For the following, take $z_1 = 1 + j$ and $z_2 = -3 + 4j$.

1. Convert z_1 and z_2 to polar and exponential notation (find r, θ).

2. Plot z_1 and z_2 on the complex plane below. Use this plane for the next questions.



3. Compute $z_1 + z_2$. Show $z_1 + z_2$. graphically on a plot in the complex plane.

4. Compute $z_1 - z_2$. Show $z_1 - z_2$ graphically on a plot in the complex plane.

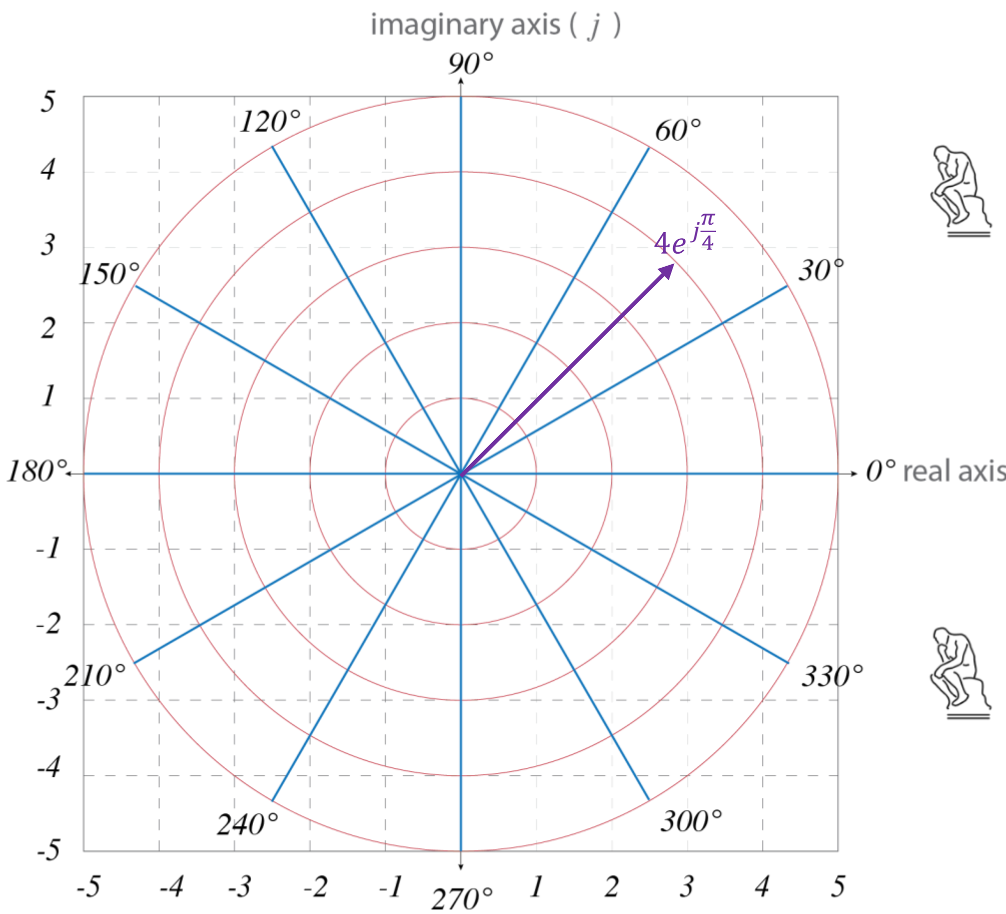
5. Compute $z_1 z_2$. Repeat the computation using a different notation.

6. Compute z_1/z_2 using complex notation. Compute z_2/z_1 and compare.

7. Compute z_1^4.

Part II: Plotting complex numbers

Complex numbers using **polar notation** are super useful for illustrating how a circuit responds to time-varying signals. The polar axes can be superimposed on the complex plane as shown:



What does $3 \cdot e^{j\pi}$ look like?

A Wavegen input signal is a sine wave, $V_{peak-peak} = 4Volts.$ In r,θ notation, what is its r ?

The **polar coordinates** (above grid of red & blue) make use of a special property of the ***exponential function*** when it operates on $j(= \sqrt{-1})$**.** You may have seen this function notated (equivalently) as:

$$e^{j\theta}, \exp(j\theta), \text{ or } e^{i\theta}$$

where θ represents an angle in radians (Recall that π radians = 180°).

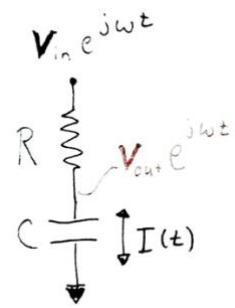
The amazing property of $e^{j\theta}$ is known as Euler's formula (section 6.3 in your book):

$$e^{j\theta} = \cos(\theta) + jsin(\theta)$$

If $\boldsymbol{\theta}$ varies with a frequency, $\boldsymbol{\omega}$, $\boldsymbol{\theta = \omega \cdot t}$, what would $\boldsymbol{e^{j\omega t}}$ look like in time?

Click this link to see. There is more info on page 6 for those who are interested.

Recall from Figure 6.3 that if we represent our cosine voltage input to a **low-pass filter** with polar notation,



$$V_{in}(t) = \boldsymbol{V}_{in} \cdot e^{j\omega t}$$

And $\boldsymbol{V}_{in}$ represents a complex number.

And remember that because the R and C are in series, the time varying current passing through both will be the same, we get,

$$\frac{V_{in}(t) - V_{out}(t)}{R} = C\frac{dV_{out}(t)}{dt}$$

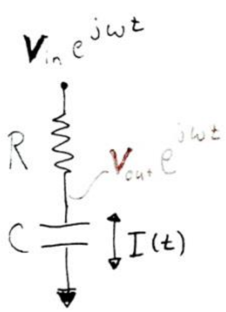
and, rearranged a bit,

$$\boldsymbol{V}_{in} \cdot e^{j\omega t} - \boldsymbol{V}_{out} \cdot e^{j\omega t} = RCj\omega\boldsymbol{V}_{out} \cdot e^{j\omega t}$$

$\boldsymbol{V}_{out}$, while a complex number, does not vary with time, so $\frac{dV_{out}}{dt}$ treats $\boldsymbol{V}_{out}$ as a constant.

Or

solving for $\frac{\boldsymbol{V}_{out}}{\boldsymbol{V}_{in}}$,



$$\frac{\boldsymbol{V}_{out}}{\boldsymbol{V}_{in}} = \frac{1}{1 + j\omega RC}$$

Let's let RC=1 second and $z_3 = \frac{1}{1+j\omega}$

And $z_4 = \frac{j\omega}{1+j\omega}$

Convert z_3 and z_4 to r, θ notation.

Plot the magnitude of *r* for z_3 and z_4 as a function of ω on a log-log scale. Let ω* vary from 10^{-3} to 10^3.

*In Matlab, you can use the command, y= logspace(-3,3), to generate a logarithmically-spaced vector, y, that spans 10^{-3} to 10^3

Knowing that z_3 and z_4 represent the $\frac{V_{out}}{V_{in}}$ of low- and high-pass filters, what do you expect the graphs to look like?

Plot θ in degrees for z_3 and z_4 as a function of ω on a semilog* scale . Let ω vary from 10^{-3} to 10^3.

*In MATLAB, use semilogx (x, y) to plot linear values for y and log x.

This plot is the phase angle part of the Bode plot. Which value should be plotted on a log scale, ω or θ?

You expect θ at the natural frequency ("cutoff frequency") to be 45°, where $\cos(\theta) = \sin(\theta)$.

Is your plot what you expect?

Watch $e^{j\omega t}$ vary as θ varies with a frequency, ω: $\boldsymbol{\theta = \omega \cdot t}$

Click this link to see.

These images illustrate for r=1,

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$

at different points in time $(\theta = \omega \cdot t)$