Complex Impedance

We have previously represented sinusoidal signals in the form of:

$$V_{in}(t) = A\sin(\omega t)$$

$$V_{out}(t) = B\sin(\omega t + \phi)$$

where B, A = amplitude
ω = frequency
ϕ = phase

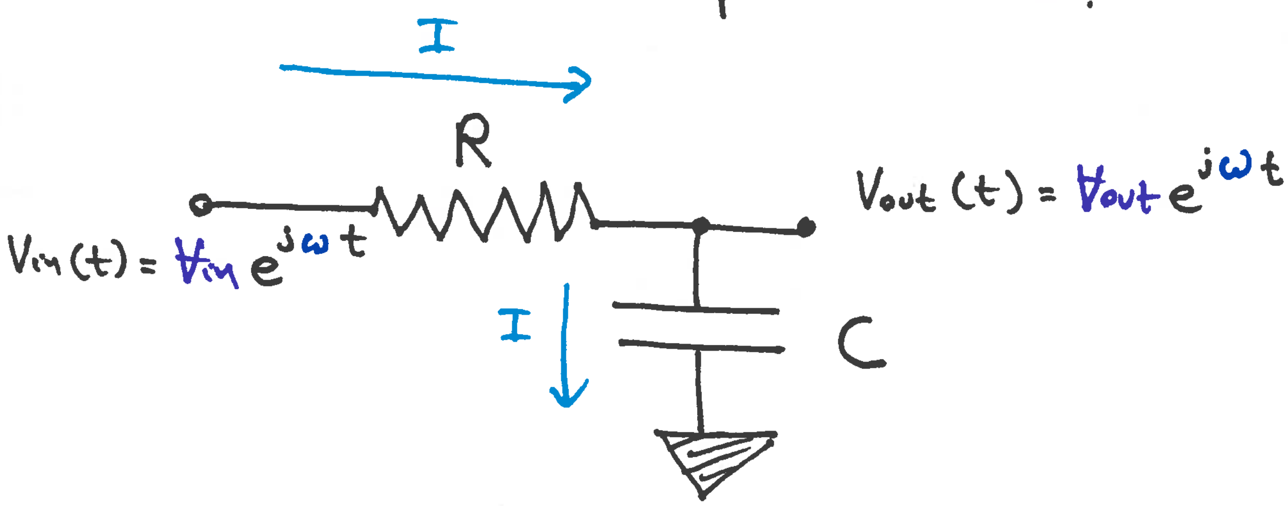
Now, we will represent voltages using the form:

$$\boxed{V(t) = \tilde{V} e^{j\omega t}}$$

where $\tilde{V}$ is a complex amplitude of the form: $\boxed{\tilde{V} = x + jy}$ or $\boxed{\tilde{V} = re^{j\phi}}$

so, $\boxed{V(t) = re^{j(\omega t + \phi)}}$

Consider the low pass filter:



$$V_{in}(t) - V_{out}(t) = IR \qquad I = C\,\frac{d\,(V_{out}(t) - 0)}{dt}$$

Setting the currents equal:

$$\frac{V_{in}(t) - V_{out}(t)}{R} = \frac{C\, d\, V_{out}(t)}{dt}$$

$$\frac{\tilde{V}_{in} e^{j\omega t} - \tilde{V}_{out} e^{j\omega t}}{R} = C j \omega \tilde{V}_{out} e^{j\omega t}$$

$$\frac{(\tilde{V}_{in} - \tilde{V}_{out}) e^{j\omega t}}{R} = C j \omega \tilde{V}_{out} e^{j\omega t}$$

$$\frac{\tilde{V}_{in} - \tilde{V}_{out}}{R} = C j \omega \tilde{V}_{out}$$

$$\boxed{\frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{1}{j\omega RC + 1}}$$

Now how about magnitude and phase?

$$\frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{1}{1 + j\omega RC} \cdot \frac{1 - j\omega RC}{1 - j\omega RC} = \frac{1 - j\omega RC}{1 + \omega^2 R^2 C^2}$$

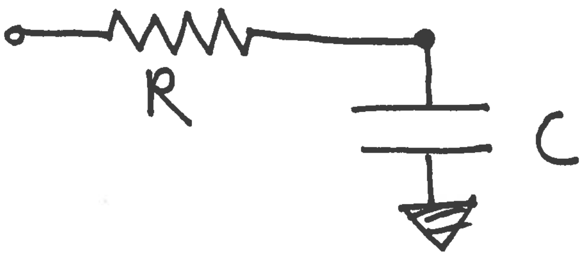
Recall we can write complex numbers in the form:

$$\frac{\tilde{V}_{out}}{V_{in}} = x + jy = re^{j\phi}$$

where $r = \sqrt{x^2 + y^2}$ and $\phi = \tan^{-1}\left(\frac{y}{x}\right)$

Consider the low pass filter:

$V_{in}(t) =$



$V_{out}(t) =$

1b + 2b

In the polar form:

$$X = \frac{1}{1+\omega^2R^2C^2} \qquad y = \frac{-RC\omega}{1+\omega^2R^2C^2}$$

Then, we have:

$$r = \left[\frac{1}{(1+\omega^2R^2C^2)^2} + \frac{R^2C^2\omega^2}{(1+\omega^2R^2C^2)^2}\right]^{1/2} = \frac{1}{\sqrt{1+\omega^2R^2C^2}}$$

$$\boxed{r = \frac{1}{\sqrt{1+(\omega RC)^2}}}$$

$$\phi = \tan^{-1}\left(\frac{\left(\frac{-RC\omega}{1+\omega^2R^2C^2}\right)}{\left(\frac{1}{1+\omega^2R^2C^2}\right)}\right) = \tan^{-1}(-RC\omega)$$

$$\boxed{\phi = -\tan^{-1}(\omega RC)}$$

So, for the low pass filter:

$$\boxed{\frac{V_{out}}{V_{in}} = \frac{1}{1+j\omega RC} = \frac{1}{\sqrt{1+(\omega RC)^2}}\, e^{-j\tan^{-1}(\omega RC)}}$$

This is the same result found using sines and cosines in Chapter 4

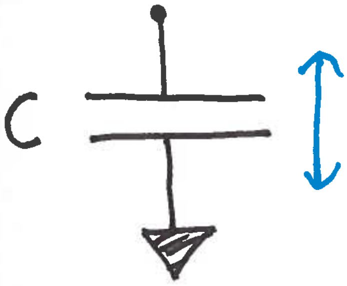
Solve for r and ϕ for:

$$\frac{V_{out}}{V_{in}} = \frac{1}{1+j\omega RC}$$

Ok, can we generalize this result?

Consider the capacitor:

$$V(t) = V_{in} e^{j\omega t}$$



$$I(t) = I e^{j\omega t} \Rightarrow$$ complex representation of current

$$I(t) = C \frac{d\,V(t)}{dt} = \frac{C\,d\,V_{in} e^{j\omega t}}{dt}$$

$$I e^{j\omega t} = C j\omega V_{in} e^{j\omega t}$$

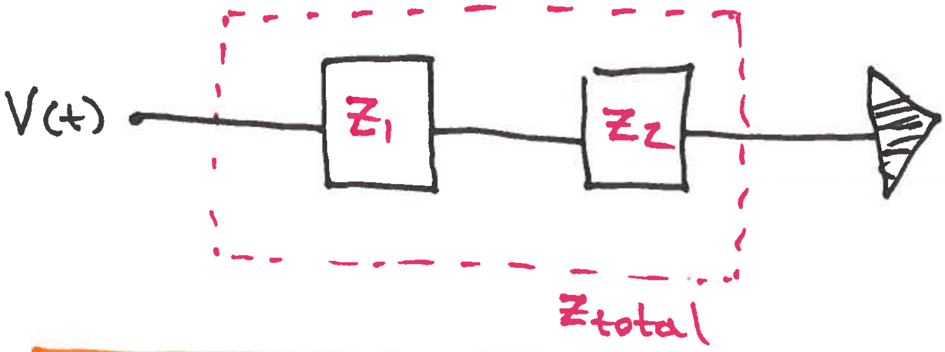
$$\boxed{\frac{V_{in}}{I} = \frac{1}{j\omega C}} \Rightarrow \text{ similar to } \frac{V}{I} = R \text{ !!!}$$

So, we can now consider the concept of an Impedance being the complex equivalent of a resistance

For a capacitor $\quad Z_C = \dfrac{1}{j\omega C}$

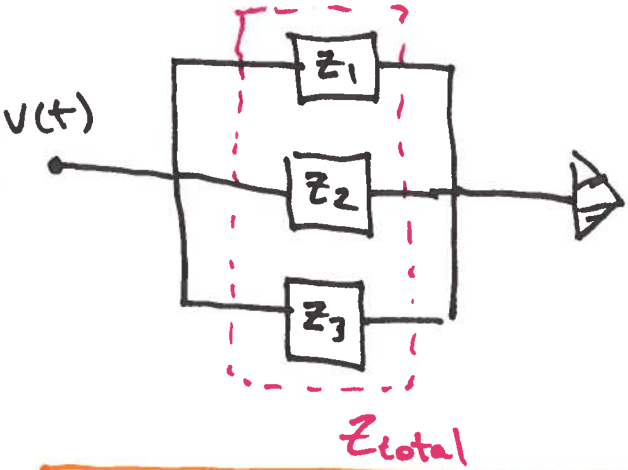
For a resistor $\quad Z_R = R$

Impedance blocks act like resistors!



$$\boxed{Z_{total} = Z_1 + Z_2 + \ldots Z_n}$$

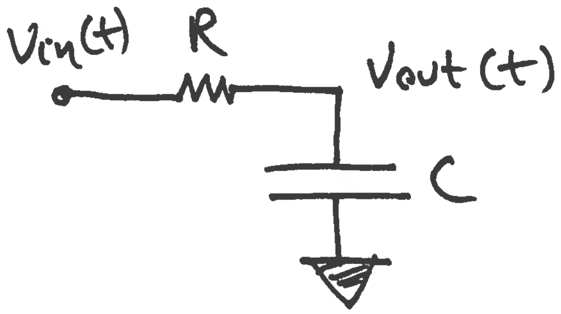
Impedances in series add!



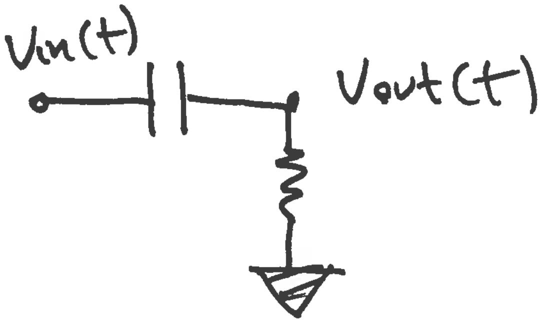
$$\boxed{Z_{total} = \frac{1}{1/Z_1 + 1/Z_2 + 1/Z_3 + \ldots 1/Z_n}}$$

Impedances in parallel act like resistors in parallel!

Use Impedances to solve the low pass filter:



Use Impedances to solve the high pass filter



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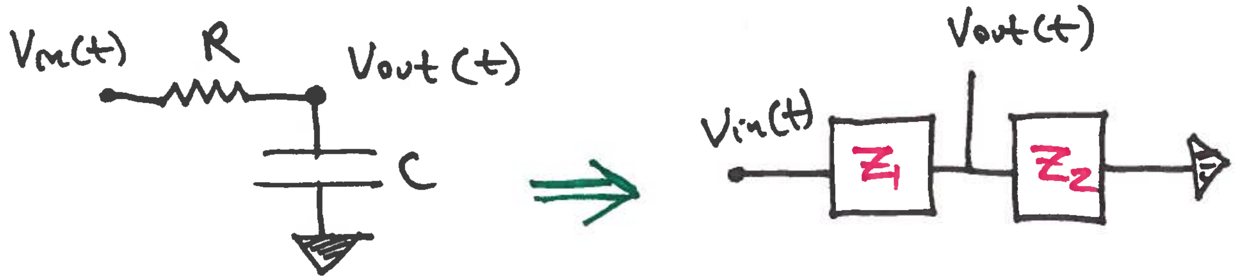
Let's apply the concept of Impedance to our high and low pass filters



$$\frac{V_{in}(t) - V_{out}(t)}{Z_1} = \frac{V_{out}(t) - 0}{Z_2}$$

$$\Rightarrow \frac{\tilde{V}_{out}\, e^{j\omega t}}{\tilde{V}_{in}\, e^{j\omega t}} = \frac{Z_2}{Z_1 + Z_2}$$

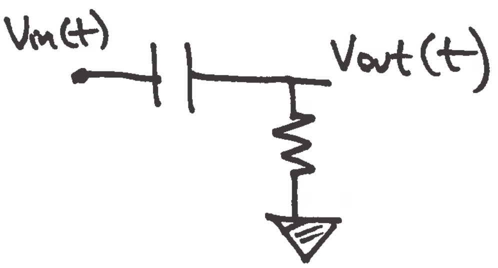
$$\Rightarrow \boxed{\frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{Z_2}{Z_1 + Z_2}}$$

This is a voltage divider!

For a low pass filter: $Z_1 = R$, $Z_2 = \frac{1}{j\omega C}$

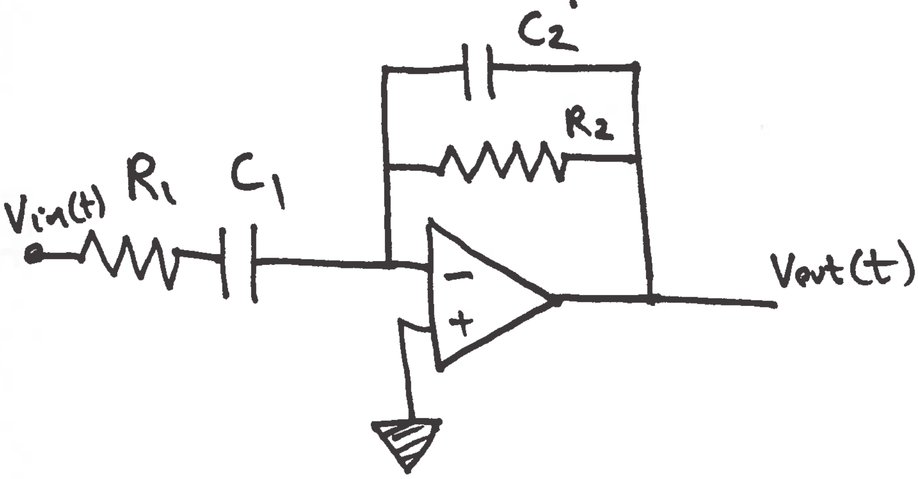
$$\frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{1/j\omega C}{R + 1/j\omega C} = \boxed{\frac{1}{1 + j\omega RC}}$$

For a high pass filter: $Z_1 = \frac{1}{j\omega C}$ $Z_2 = R$

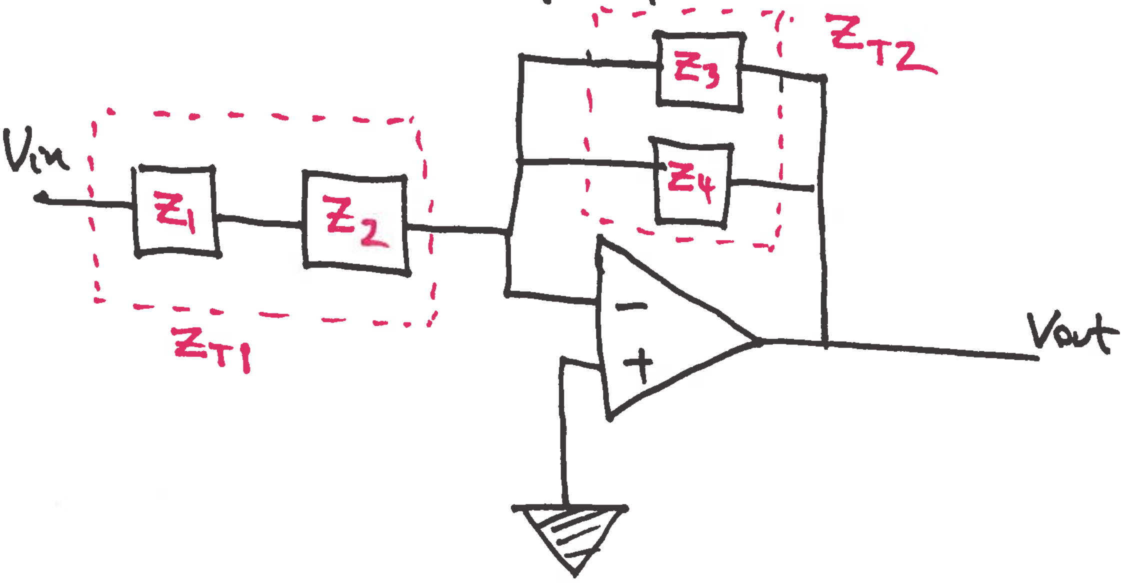


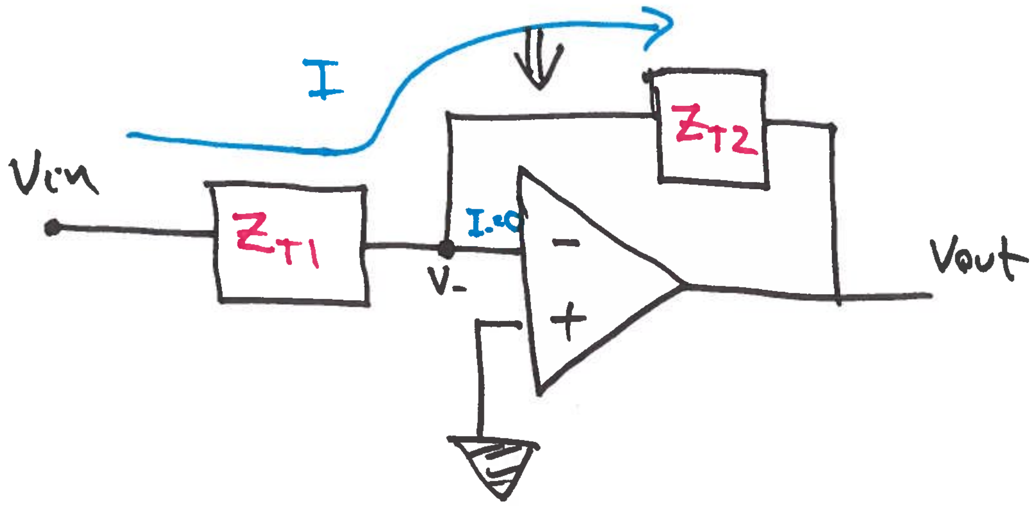
$$\boxed{\frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{Z_1}{Z_1 + Z_2} = \frac{j\omega RC}{1 + j\omega RC}}$$

Consider the bandpass filter:



$\Downarrow$ represent with impedances





$$Z_{T1} = Z_1 + Z_2 = R_1 + \frac{1}{j\omega C_1}$$

$$Z_{T2} = \frac{1}{1/Z_3 + 1/Z_4}$$

$$Z_{T1} = Z_1 + Z_2 = R_1 + \frac{1}{j\omega C_1}$$

$$\boxed{Z_{T1} = R_1 + \frac{1}{j\omega C_1}}$$

$$Z_{T2} = \frac{1}{\frac{1}{Z_3} + \frac{1}{Z_4}} = \frac{R_2}{1 + j\omega C_2 R_2}$$

$$\boxed{Z_{T2} = \frac{R_2}{1 + j\omega C_2 R_2}}$$

Now, Apply the op-amp rules : $I_- = 0$

$V_{in}(t) - V_- = I\, Z_{T1}$ and $V_- - V_{out}(t) = I\, Z_{T2}$

With the reference to ground, $V_- = 0$

$$\frac{V_{in}(t) - 0}{Z_{T1}} = \frac{0 - V_{out}(t)}{Z_{T2}}$$

$$\boxed{\frac{V_{out}(t)}{V_{in}(t)} = \frac{\mathbf{V}_{out}}{\mathbf{V}_{in}} = -\frac{Z_{T2}}{Z_{T1}}}$$

The analysis using impedance blocks is super quick!

We can now substitute what we found for Z_{T1} and Z_{T2} and learn more about this circuit

$$\frac{V_{out}}{V_{in}} = \underbrace{\frac{-R_2}{1+j\omega C_2 R_2}}_{Z_{T1}} \cdot \underbrace{\frac{1}{R_1 + \frac{1}{j\omega C_1}}}_{Z_{T2}}$$

$$= \frac{-R_2}{1+j\omega C_2 R_2} \cdot \frac{j\omega C_1}{1+j\omega C_1 R_1}$$

$$\boxed{\frac{V_{out}}{V_{in}} = \frac{-R_2}{R_1} \cdot \frac{1}{1+j\omega C_2 R_2} \cdot \frac{j\omega R_1 C_1}{1+j\omega R_1 C_1}}$$

What does this result tell us about the circuit

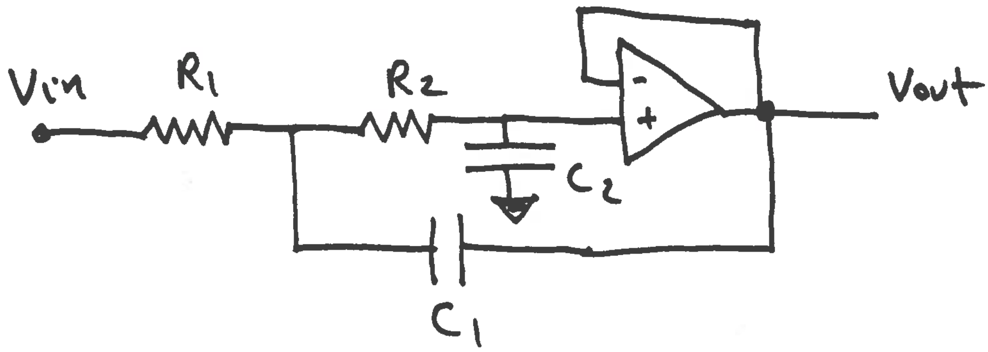
$$\frac{V_{out}}{V_{in}} = \frac{-R_2}{R_1} \cdot \frac{1}{1+j\omega C_2 R_2} \cdot \frac{j\omega R_1 C_1}{1+j\omega R_1 C_1}$$

Amplifier with gain $\frac{R_2}{R_1}$

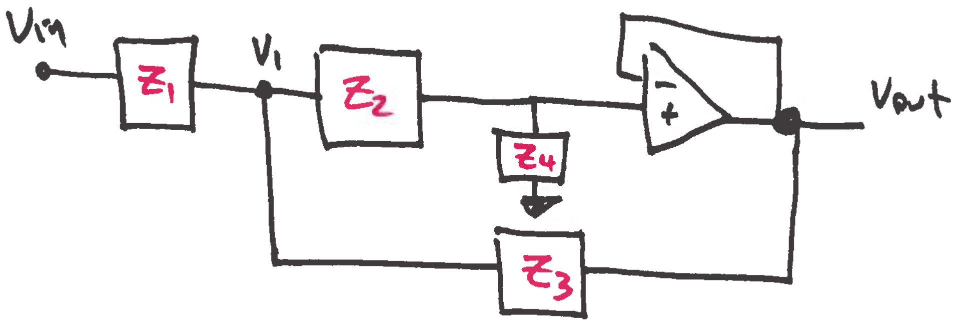
Low pass filter $\omega_c = \frac{1}{R_2 C_2}$

High pass filt[er] $\omega_c = \frac{1}{R_1 C_1}$

Sallen-Key Topology



$\Downarrow$



$$\underline{\underline{V_+ = V_- = V_{out}}}$$

Applying op-amp rules:

$$\frac{V_{in} - V_1}{Z_1} = \frac{V_1 - V_{out}}{Z_3} + \frac{V_1 - V_{out}}{Z_4}$$

and

$$\frac{V_1 - V_{out}}{Z_2} = \frac{V_{out}}{Z_4}$$

$$\boxed{\frac{V_{out}}{V_{in}} = \frac{Z_3 Z_4}{Z_1 Z_2 + Z_3(Z_1 + Z_2) + Z_3 Z_4} \quad \text{let } \begin{matrix} Z_1 = Z_2 \\ Z_3 = Z_4 \end{matrix} \quad = \frac{Z_3^{\,2}}{(Z_1 + Z_3)^2}}$$

$$\frac{V_{out}}{V_{in}} = \frac{(1/j\omega C)^2}{(R + 1/j\omega C)^2} \implies \text{2nd order low pass}$$