

Resistance

I. HYDRAULIC SYSTEM

Let's start with a simple physical system that you might have some intuition about. Imagine a tank of water and a pipe. One end of the pipe is connected to the tank and the other end is open to the atmosphere on the right. A schematic is shown in Figure 1. The tank has a constant cross sectional area, A , and the water height is H . The pipe has diameter D and length L . The volumetric flow rate (measured in liters per second, for example) through the pipe is Q . The system is sized such that the volumetric flow rate is small compared to the volume of the tank so the height of the water changes very slowly as water drains.

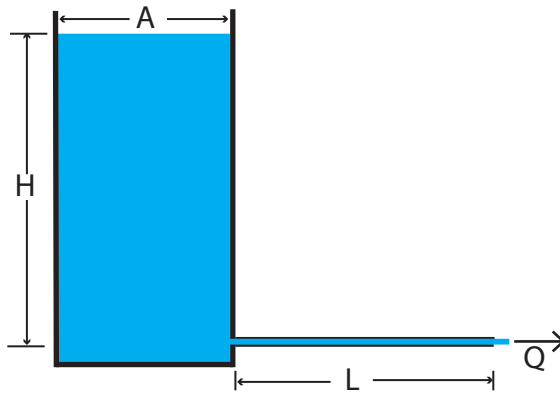


FIG. 1 Schematic of a simple hydraulic system. A tank of water is drained by a pipe. We want to understand the relationship between the volumetric flow rate and the height of the water.

In this system the high pressure of the water near the bottom of the tank is what drives the flow. The pressure, P , is force per unit area and SI units for pressure are pascals (Pa), or newtons per square meter (N/m^2). At the bottom of the tank the pressure is proportional to the height of the water and is given by $P = \rho g H$ where ρ is the density of water ($1000 \text{ kg}/\text{m}^3$), g is $9.8 \text{ m}/\text{s}^2$, and H is the height of the water in meters. If you forget this formula, it is easy to derive. The total force due to the weight of water in the tank is $F = mg$ or equivalently, $F = \rho A H g$ since AH is the volume of water in the tank. This total force is distributed over the bottom of the tank and must be balanced by the tank pushing back. Therefore the force per unit area on the bottom, the pressure, is $F/A = P = \rho g H$.

A. Resistance

A natural question to ask is what is the volumetric flow rate out of the tank? You would probably guess that the flow rate would depend upon the height of water in the tank. To answer the question experimentally, all you need is a beaker to measure volume and a stop watch. For small system, we conducted an experiment in the kitchen with two different length of pipes. This experiment is simple enough that anyone can do it. The data in Figure 2 were collected by an 8 and 5 year old. They measured the time required to fill a 10 ml beaker with a stop watch for different heights of water in the tank.

We find that for the two experiments in Figure 2 the relationship between water height and flow rate seems to be linear. The slope can be determined from the experimental data. In this experiment we find that the slope depends upon the length of the pipe; a longer pipe has a steeper slope. We can guess that the slope would also depend upon the pipe's diameter. We expect that the slope would be steeper for a thinner pipe. A thinner pipe will have a lower flow rate for the same pressure.

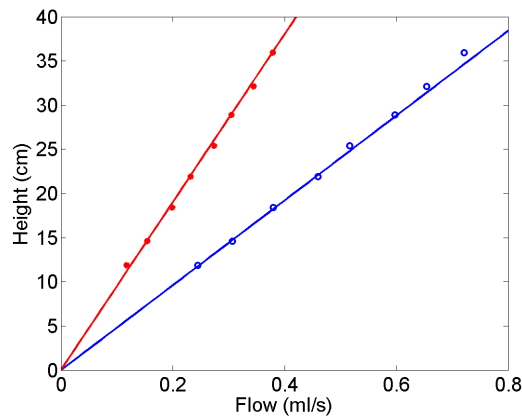


FIG. 2 Experimental data for pipe of 1/16 inches (1.6 mm) in diameter and two different lengths. The data in red with the solid stars has a pipe with twice the length ($L = 1.4$ m) of the data with the data in blue with the open circles ($L = 0.7$ m). The points are experimental data and the solid line is an approximate linear fit to the data. Special thanks to Charlotte (8) and Luke (5) for collecting this experimental data.

Now it is the tank pressure, not really the water height, that provides the physical mechanism responsible for the flow. So we are led to conclude that the behavior of our system could be described by,

$$P = QR.$$

The constant R is called the *resistance*. When resistance is high, a large pressure is needed to drive a small flow rate. In the hydraulic case, it turns out that sometimes the resistance is not a constant and can depend upon the flow rate itself. For our experiments, we measured a constant resistance and we will assume that is the only case of interest.

It is important to realize that the pressure we use is really the pressure difference applied *across* the pipe, $\Delta P = P_{inlet} - P_{outlet}$. To be a more precise we should write,

$$\Delta P = QR,$$

where ΔP implies the change in pressure from the pipe's inlet to outlet. As far as the water flow is concerned, the overall pressure is unimportant - it is the difference from inlet to outlet. In this simple example, the applied pressure drop is $\Delta P = \rho g H$ since the atmosphere acts equally on the water in the tank and the pipe exit.

B. Resistors in series

In Figure 2, one set of data was taken for a pipe twice the length of the other. We can think about the longer pipe as taking two equal lengths of pipe and adding them in *series*. If we carefully look at the data, we find that the resistance of two pipes in series is exactly twice that of the single pipe.

We can understand this result by looking at a more general case of adding pipes of different resistance in series, shown schematically in Figure 3. The pressure drop across each section would simply add to equal the total pressure applied across both sections of pipe,

$$\Delta P = \Delta P_1 + \Delta P_2.$$

Substituting the relationship for pressure and flow for each if the two pipes we have,

$$\Delta P = Q_1 R_1 + Q_2 R_2.$$

When two pipes are added in series, they must have the same flow rate through them - the water cannot change its volume and there is nowhere else for water to go. Therefore $Q_1 = Q_2 = Q$. We can rewrite the overall pressure-flow relationship as

$$\Delta P = QR_1 + QR_2 = Q(R_1 + R_2).$$

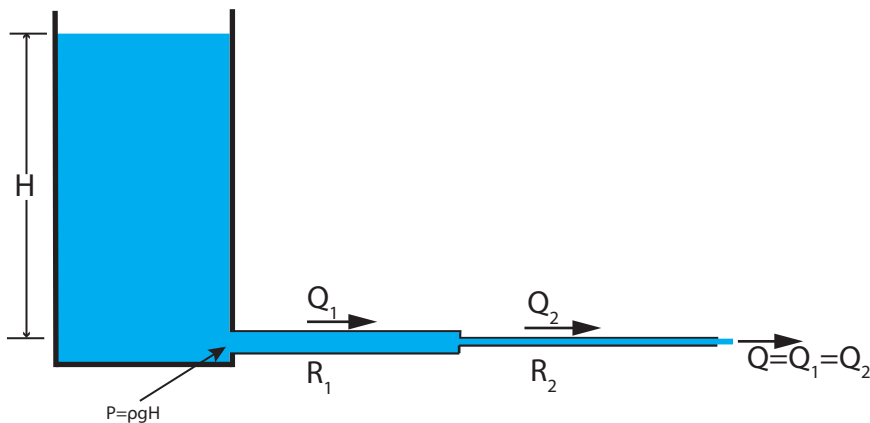


FIG. 3 Schematic of a simple hydraulic system with two pipes in series.

Whenever we have two pipes in series, we simply add the resistances to get the total resistance to flow out the tank. Notice that if one of the pipes has a much large resistance than the other, then effective resistance would be just a little bit higher than larger of the two resistances. Remember this point; if two you have two resistors in series and the resistances are very different sizes, the effective resistance is close to that of the *largest* of the two resistors.

We could generalize this expression even further and would find that if we added several pipes in series, the total resistance would equal the sum of the individual resistances;

$$R = R_1 + R_2 + R_3 \dots \quad \text{Series resistance}$$

C. Resistors in parallel

Now imagine we take the same tank and put two pipes out as shown in Figure 4. In this case, the total flow out of the tank would be the sum the flow out of each independent pipe,

$$Q = Q_1 + Q_2.$$

Substituting the relationship for pressure and flow for each if the two pipes we have,

$$Q = \frac{\Delta P_1}{R_1} + \frac{\Delta P_2}{R_2}.$$

The pressure applied across the two pipes is the same, thus

$$Q = \Delta P \left(\frac{1}{R_1} + \frac{1}{R_2} \right).$$

Rearranging this expression to the usual form we have,

$$\Delta P = Q \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = QR.$$

where the total effective resistance of the two pipes is

$$R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}.$$

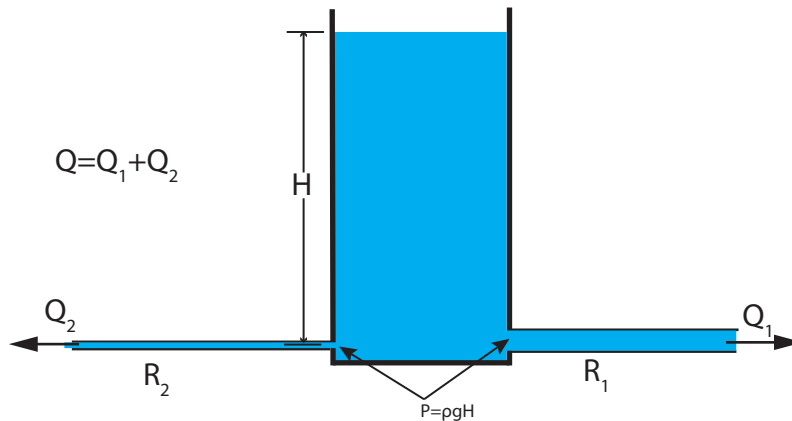


FIG. 4 Schematic of a simple hydraulic system with two pipes in parallel.

In the case where $R_1 = R_2$ the effective resistance would become $R = R_1/2$. This result makes sense because if we take two equal pipes draining the tank, the flow rate out would double (or the resistance would be halved) from the case of a single pipe. We can also rewrite the effective resistance expression as,

$$R = \frac{R_1}{1 + \frac{R_1}{R_2}}$$

This form let's us see that if $R_1 \ll R_2$ then the effective resistance is just a bit lower than R_1 . With resistors in parallel, if the two resistors are very different sizes then the effective resistance is close to the *smallest* of the two resistors.

The result above would easily generalize to more than two pipes draining the tank,

$$R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \dots} \quad \text{Parallel resistance}$$

II. CIRCUITS - ELECTRICAL RESISTANCE

These concepts and equations carry over to the analysis of our first passive circuit element, the resistor. A picture of a resistor of the style we will use in this course and the symbol used in circuit drawings is shown in Figure 5. Like a pipe, a resistor works equally as well which ever way it is oriented in a system; there is no positive or negative side. Resistors used in modern electronics are much smaller than the ones we use, but they work the same way. The basic equation for the resistor known as Ohm's law,

$$\Delta V = IR.$$

Here, ΔV is voltage measured in volts, I is current measured in amps, and R is the resistance in ohms (Ω).

Making the analogy to the hydraulic example, voltage is like pressure, current is like volumetric flow rate, and electrical resistance is like the pipe's resistance. Just like with pressure, it is the voltage *difference* across the resistor that we use in Ohms law. The values of voltage that we will typically use to power circuits in the course are 5 Volts. This is value is common in many electronic devices and is compatible with running a device off battery power.

Electrical current is measured in amperes or amps for short. The ampere is equivalent to a coulomb per second. A coulomb is the unit of electric charge and is equivalent to the charge on 6.24×10^{18} electrons. The current, I , flowing through a resistor is the amount of charge per unit time passing through. Just like the flow of water, current flows through a circuit in a conserved way. For any part or node in a circuit, the amount of current flowing in must equal

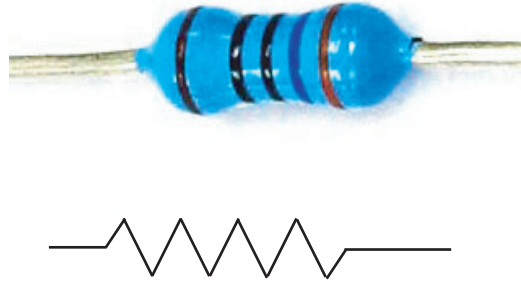


FIG. 5 Picture of a resistor of the style we will use in this course. The diagram shows the schematic symbol used in circuit drawings.

the amount of current flowing out. The analogy with the water flow in the pipe is a good one. Current can be thought of as a flow of charge.

Just like resistance of a pipe can change depending on the diameter and length, the resistance of an electrical resistor depends on its size, material, and how it is made. Just like with a pipe, resistance increases linearly with the length and increases inversely with the cross section area. While resistors come in all shapes and sizes for different reasons, the physical form factor of resistors we use in lab will typically look like those in Figure 5. The different values of resistance depend upon the details when the resistor was manufactured, but resistors are designed for a particular value. Resistance in practical circuits can span many orders of magnitude. In this course we will use resistors ranging from $10\ \Omega$ to $10,000,000\ \Omega$. The range of resistors that one can purchase is much wider than this. We use the kilo and mega prefixes to denote the size; a $1,000\ \Omega$ resistor would be 1 kilo-ohm or $k\Omega$ and a $1,000,000\ \Omega$ resistor would be 1 mega-ohm or $M\Omega$. In class when we are speaking, we will usually refer to these values as “1 K” and “1 Meg”. In lab, we typically use 1 % resistors meaning the manufactured value is specified within 1 percent. One can buy higher precision if you need it. The style resistor we use costs around 1 cent each and the small ones found in modern electronics are typically much less than this. Resistors are inexpensive components.

A. Resistors in series and parallel

The rules we derived for the pipe for resistances in series and parallel work equally as well here. A circuit with two resistors connected to a constant voltage source such as a battery or power supply is shown in Figure 6. On the left figure the two resistors are in series and on the right they are in parallel. Just as with the hydraulic system, we want to come up with a relationship between the applied voltage drop, V , and the resulting total current, I , through the two resistors. There are two circuit symbols used in the schematic. One is the resistor that we discussed already. The symbol with the two lines, one longer than the other represents a voltage source. Think of this as a battery where the long side is the positive terminal of the battery and the short side is the negative terminal. The voltage across the battery is, V . If it were a real battery, V would equal the voltage written on the side of the battery such as 1.5 volts for a AA battery. Since the two terminals of the battery are connected across the resistors, this total voltage difference is applied across the resistors.

When we have resistors in series, the total voltage drop across two resistors is set by the battery and is equal to the sum the voltage drops across each resistor individually,

$$V = \Delta V_1 + \Delta V_2.$$

The current flowing through the two resistors, just like the water flow, must be the same. Since charge flows through the circuit in a conserved way, $I_1 = I_2 = I$. Using Ohm’s law in our above expression for V we obtain,

$$V = I_1 R_1 + I_2 R_2 = I(R_1 + R_2),$$

which was identical to what we derived for the water flow. The battery sees the two individual resistors as an equivalent resistance which is simply the sum.

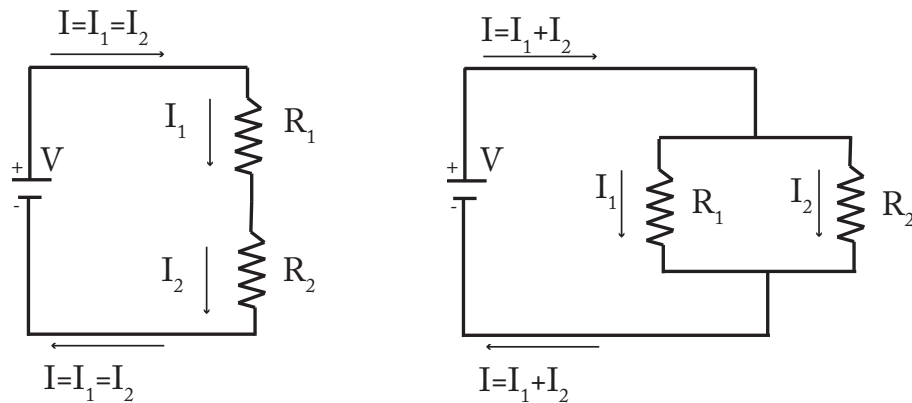


FIG. 6 Schematic of two resistors connected to a constant voltage source, V . On the left the resistors are in series and on the right the resistors are in parallel.

When we have resistors in parallel, the total current from the battery must equal to the sum of the current flowing through the two resistors,

$$I = I_1 + I_2.$$

Using Ohm's law,

$$I = \frac{\Delta V_1}{R_1} + \frac{\Delta V_2}{R_2}.$$

However, the voltage drop across each resistor is the same and is set by the battery, therefore,

$$I = V \left(\frac{1}{R_1} + \frac{1}{R_2} \right),$$

or

$$V = I \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}.$$

The battery sees the same current as though there were a resistor with an equivalent resistance of $R = 1/(1/R_1 + 1/R_2)$. The equivalent resistance of two resistors in parallel is exactly as we found previously in the hydraulic case.

B. Voltage divider

Let's look at a simple example of two resistors in series and ask what the voltage is between the two resistors. The applied voltage (from a battery or power supply) is V_{in} . The voltage between the resistors is V_{out} . We take the voltage at the negative terminal of the battery to be zero. Since we care only about voltage differences, we are allowed to set our reference for zero anywhere we like. The circuit schematic is shown in Figure 7. The total current flowing through the two resistors is found by Ohm's law with the effective series resistance,

$$I = \frac{V_{in} - 0}{R_1 + R_2}.$$

Using Ohm's law for the second resistor,

$$V_{out} - 0 = IR_2.$$

Putting the previous two equations together we have,

$$V_{out} = V_{in} \frac{R_2}{R_1 + R_2}.$$

If $R_2 \gg R_1$ then $V_{out} \approx V_{in}$. If $R_2 \ll R_1$ then $V_{out} \rightarrow 0$. The above circuit is known as a voltage divider because it, well, divides the input voltage V_{in} .

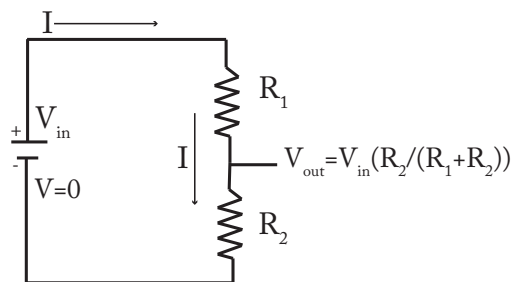


FIG. 7 Voltage divider circuit.

III. KIRCHHOFF'S CIRCUIT LAWS

In the previous section we derived laws for resistors in series and in parallel. Generalizing some of the ideas we have already used leads us to Kirchhoff's laws, which are attributed to Gustav Kirchhoff in 1845.

Kirchhoff's current law (KCL) states that the sum of the currents flowing into any circuit node must equal the sum of the currents flowing out. The law is based on conservation of charge. We already used this law when we analyzed resistors in series and parallel.

Kirchhoff's voltage law (KVL) states that the directed sum of the voltage differences around any closed loop in a network is zero. The only tricky part about KVL is keeping the signs straight. For example, as we sum the voltages around a loop, we count a resistor voltage drop as positive if we are summing in the direction of the current. The resistor voltage drop is counted as negative if we are summing in a direction against the current.

Let's do a simple example of resistors in parallel, shown in Figure 8. First let's draw an assumed direction of the current flow on the diagram. Next, we assign a label to the current in each branch of your network. Since this example

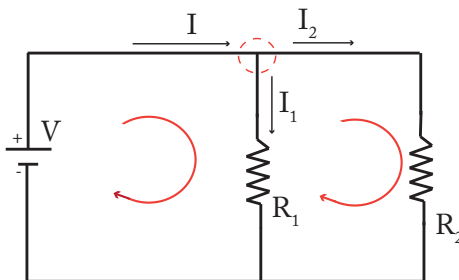


FIG. 8 Resistors in series analyzed using KCL and KVL. The red dashed circle shows the node in this circuit. The two red arrows indicate the direction we selected to sum the voltage drops around the two loops. The arrows denote our assumed direction of current flow.

has so few branches, it is not hard to guess the correct direction of current. However, note that if we guessed the wrong direction these current would come out as negative currents at the end of the calculation. The analysis works whether you actually guess the direction correctly or not.

Now let's apply KCL. There are only two nodes in this circuit and each node tells us the same information. The sum of the currents at the upper node tells us the current in equals the sum of the current out. KCL tells us,

$$I = I_1 + I_2$$

Now let's apply KVL to the two loops of the circuit. In Figure 8 we draw the direction which we will sum the voltage drops. The outer loop with the two resistors would tell us,

$$\Delta V_2 - \Delta V_1 = 0$$

The sign of the voltage drop across resistor 2, ΔV_2 , is positive because the summing direction (denoted by the red arrow) and the assumed current flow are going in the same direction. The sign of the voltage drop across resistor 1, ΔV_1 , is negative because the summing direction and the assumed current flow are going in opposite direction. Combining Ohm's law with KVL around this loop yields,

$$I_2 R_2 = I_1 R_1 \quad \rightarrow \quad I_2 = I_1 \frac{R_1}{R_2}.$$

Applying KVL to the inner loop yields,

$$\Delta V_1 - V = 0.$$

Here the sign of the voltage drop across resistor 1 is positive because the summing direction and the assumed current flow are going in the same direction. The sign of the drop at the voltage source, V , is negative because we pass from the negative to the positive terminal of the battery as we progress in the summing direction. Combining KVL for this loop along with Ohm's law yields,

$$V = I_1 R_1.$$

We now have three equations for three unknown currents, I_1 , I_2 , and I . Combining the first two equations gives,

$$I = I_1 + I_2 = I_1 \left(1 + \frac{R_1}{R_2} \right) \rightarrow I_1 = I \frac{1}{1 + \frac{R_1}{R_2}}$$

Which upon using KVL from the inner loop gives,

$$V = I_1 R_1 = I \frac{R_1}{1 + \frac{R_1}{R_2}} = I \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}.$$

We obtained the same result as before. When doing circuit analysis, we will always invoke KCL. We did this when we first derived resistors in series and parallel. In simple circuits you will often find that you can get at the result you want without explicitly calling out KVL as we did in our early examples. However, KVL is very useful as it provides a systematic way of solving complex circuits.