Capacitance

I. PHYSICAL SYSTEM

Imagine a tank of water connected to a pipe at the bottom where the other end of the pipe is open to the atmosphere. The tank has a constant cross sectional area, A, and water height H. Water flows through the pipe from the left to the right until the tank is empty. The volumetric flow rate through the pipe is Q. In this system the pressure at the bottom of the tank is given by $P = \rho g H$ where ρ is the density of water (1000 kg/m^3), g is 9.8 m/s^2, and H is the water height (m).
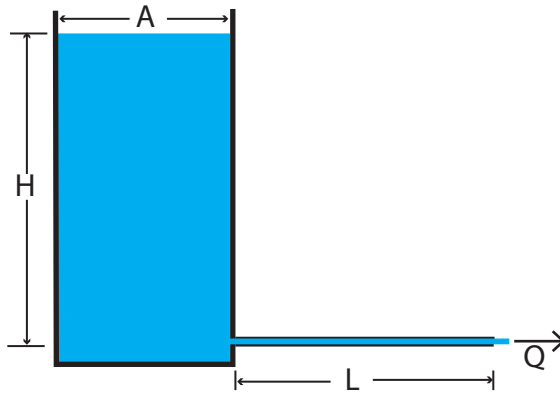

FIG. 1 Schematic of a simple hydraulic system.

Previously, we used the same physical system to introduce the idea of resistance to flow. In the notes on resistance, we described a case where we measure a linear relationship between the height of the water in the tank and the flow rate out of the tank. Our empirically validated hydraulic version of Ohm's law was,

$$\Delta P = RQ,$$

where ΔP was the pressure applied across the pipe, Q is the volumetric flow rate of water through the pipe, and R is the hydraulic resistance. The resistance depends upon the size of the pipe and is something we can easily measure.

II. CAPACITANCE - HYDRAULIC SYSTEM

Let's consider the case where we fill the tank and monitor the water height as a function of time as the tank drains. As the water flows out and the tank empties, the pressure applied across the pipe decreases. As the pressure decreases, so does the flow rate. Since the rate that the water level changes is proportional to the flow rate, the rate of height decreases begins to slow. Experimental data were recorded by simply timing the water height at different levels with a stopwatch and the result is shown in Figure 2.

Let's see if we can use some simple ideas to understand the behavior of our data. The rate of change of the volume of fluid, $V(t)$, contained in the tank is equal to the flow rate through the tube,
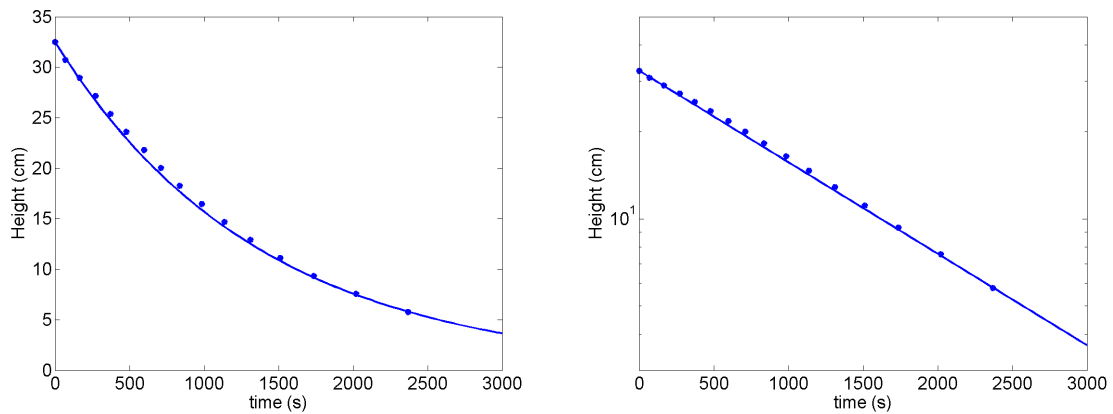
$$\frac{dV(t)}{dt} = -Q(t).$$

FIG. 2 Behavior of tank draining problem. The points are experimental measurements using nothing more than a stopwatch and the solid curve is from the theoretical prediction. On the left the y-axis is on a linear scale and on the right the y-axis is logarithmic.

Note that we are explicitly stating that the volume of fluid in the tank and the flow rate out are functions of time. The negative sign denotes that the height decreases when the flow is from left to right, defined as a positive flow rate. Since the cross sectional area of the tank is constant we can recast our expression in terms of the water height,

$$A\frac{dH(t)}{dt} = -Q(t).$$

The pressure drop across the pipe is related to the height of water in the tank,

$$\Delta P(t) = \rho g H(t).$$

We also found last week that in our system we have a constant resistance such that the flow and pressure are related as,

$$\Delta P(t) = Q(t)R.$$

Therefore the flow rate is related to the height as,

$$Q(t) = \frac{\rho g H(t)}{R}.$$

Substituting this result into our rate equation yields,

$$A\frac{dH(t)}{dt} = -\frac{\rho g}{R}H(t).$$

Let's study this expression for a bit. Ignore the factors of ρ and g for now. Since we are talking about water and doing the experiments on earth, there is not much we can do about these parameters. We can change R by selecting a different tube. A higher resistance will decrease the rate that the tank height falls. The parameter A is the size of the tank. If the tank has a lot of *capacity*, the tank's cross sectional area is large. When the tank has a large *capacity* then the height changes slowly. We can rearrange the equation to give,

$$\frac{dH(t)}{dt} = -\frac{\rho g}{RA}H(t) = -\frac{H(t)}{\tau}.$$

Note that the parameter, $\frac{AR}{\rho g}$, must have units of time. We call this the time constant for the system, $\tau = \frac{AR}{\rho g}$. Note that the time constant is related to the product of the tube resistance and the tank's capacity.

In order to solve this simple differential equation, we separate the variables,

$$\frac{dH(t)}{H(t)} = -\frac{dt}{\tau},$$

and integrate once,

$$\ln(H(t)) = -\frac{t}{\tau} + C,$$

where C is a constant of integration. We find the constant by using the initial state of the system, namely we know the initial height, $H(t = 0) = H_0$,

$$\ln(H_0) = -\frac{0}{\tau} + C.$$

Therefore using the now know value of C,

$$\ln(H(t)) = -\frac{t}{\tau} + \ln(H_0).$$

This result explains why the data were linear on the semilog plot in Figure 2. Taking the exponential of both sides,

$$H(t) = \mathrm{e}^{-t/\tau + \ln(H_0)} = \mathrm{e}^{\ln(H_0)}\mathrm{e}^{-t/\tau} = H_0\mathrm{e}^{-t/\tau},$$

The final result is thus,

$$H(t) = H_0\mathrm{e}^{-t/\tau}.$$

The height falls exponentially and the time scale on which it does so is τ.

In the previous experiment on resistance we measured R for a few different tubes. Using this measured value of R and the cross sectional area of the tank our value of τ for the experimental measurements was ~ 1373 seconds. The predicted curve is shown as the solid curve in Figure 2 and we see quite excellent agreement. Note in Figure 2 that we show the data on linear and logarithmic coordinates on the y-axis. Note that exponential decay plotted with a log axis in y yields a line.

III. EXPONENTIALS

The exponential function is a wonderful function. It shows up in a a number of systems from hydraulic, electrical, mechanical, chemical, thermal, biological, financial, and many more. Even though it is likely you have encountered this function before, it is important to make sure you understand a few properties.

Let's just plot the function,

$$y(t) = \mathrm{e}^{-t/\tau}.$$

The time constant, τ, determines the characteristic time for decay. In Figure 3 we show a time constant of $\tau = 1$, $\tau = 2$ and $\tau = 6$. If one draws a straight line from the initial value at $t = 0$ to the value of τ on the time axis, this straight line will have the same slope as the exponential at $t = 0$. This linear extrapolation is shown as the dashed line in Figure 3. Using this type of extrapolation of the slope is an excellent way to estimate the time constant from experimental data. Note that when $t = \tau$ the function has the value of $1/\mathrm{e} \approx 0.37$. The value of $y = 1/\mathrm{e}$ is shown as the dotted line in Figure 3.

Another trick is that instead of plotting the x axis of our figures as time in units of seconds, we could plot the time axis in units of τ. This change might sound odd, but is no different than changing our units from seconds to minutes to hours. When we plotted the exponential function in these time units all the curves would look the same. Therefore, any system with exponential behavior has the same basic function, it can just be stretched out in time or in it's vertical scale.

IV. CAPACITORS IN CIRCUITS

After the resistor, the capacitor is the next passive electrical component we will study. Physically, a capacitor can be created from two parallel plates separated by a thin nonconducting layer - such as a gap of air or a polymer film. The symbol on a circuit schematic is two parallel lines, representative of the capacitors physical nature. A capacitor can act as an energy storage device where the capacitor stores the energy in the electrical field between the two plates. A detailed description of the mechanism behind the capacitors behavior is probably best left for another day.
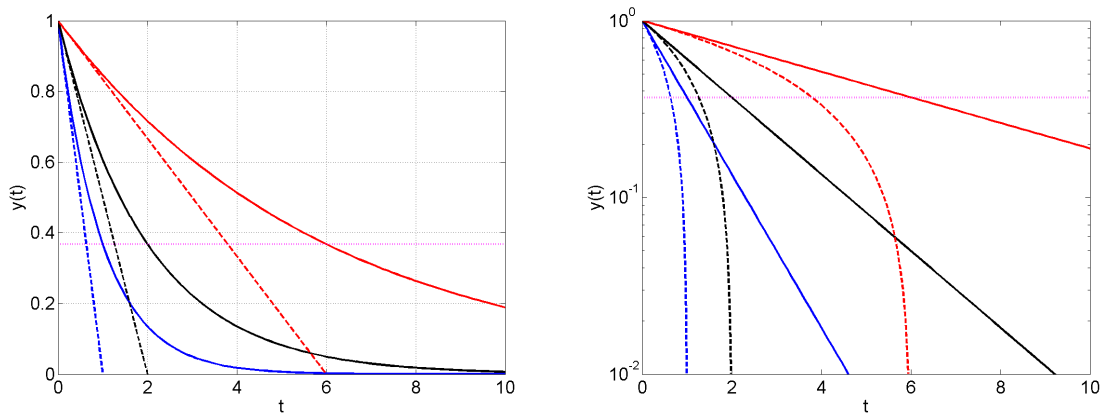
FIG. 3 Effect of an increasing time constant on the resulting dynamics of exponential decay. Here we show results for $y = e^{-t/\tau}$ for $\tau = 1, 2, 6$ as the solid lines. The dashed lines are for the initial linear extrapolation. On the left the y-axis is linear and on the right it is logarithmic.

Suppose we wanted to simply characterize the capacitors voltage-current relationship. We could conduct a number of experiments and would quickly determine that the capacitors behavior is time dependent - it is not as simple as the resistor which provides an instantaneous relationship between voltage and current. A set of careful experiments would reveal to us that the law should be written as

$$C\frac{d\Delta V(t)}{dt} = I(t).$$

Here, I is the current, C is the constant capacitance, and ΔV is the voltage drop across the capacitor. The value of C depends upon the component For the time being, just like Ohm's law, let's take this relationship at face value and note that we will verify the relationship through our observations in the lab.

The numerical value of C has units of farads. One farad is equal to one coulomb of charge accumulated when the capacitor is charged to one volt. Capacitors come in all sorts of form factors and physical sizes. Like resistors, the values of the capacitance can vary by many orders of magnitude. The smallest capacitors we will use in this course are about 10 picofarads (pf). A picofarad is 10^{-12} farads. The largest we will use are about 100 microfarads (μf) or 0.0001 farads. A one farad capacitor is really large.

V. RC CIRCUITS

Consider a resistor and capacitor arranged in series as shown in Figure 4. The voltage at the left side of the resistor will be set by some external source and will be the input voltage, V_{in}. The voltage between the resistor and capacitor will be considered the output of the circuit, V_{out}. Initially $V_{\text{out}} = V_{\text{in}} = 1$. When the voltage is set to one at these two points, no current is initially flowing through the system. At $t > 0$ the input voltage is instantaneously pulled down to $V_{\text{in}} = 0$.

The current through the resistor is given by Ohm's law. Since Ohm's law requires that we use the voltage drop across the resistor, we have,

$$I_R(t) = \frac{V_{\text{in}} - V_{\text{out}}(t)}{R}.$$

At $t > 0$, $V_{\text{in}} = 0$ and therefore the current is given as

$$I_R(t) = \frac{-V_{\text{out}}(t)}{R},$$

where we explicitly remind ourselves for now that the current and output voltage are time dependent.

Using our law for the capacitor, the current through the capacitor is given as ,
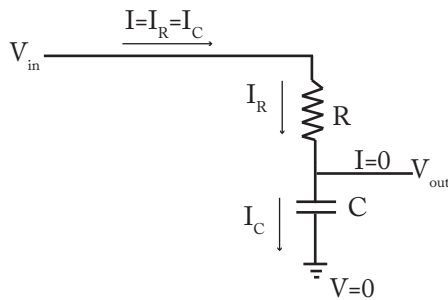
$$C\frac{dV_{\text{out}}(t)}{dt} = I(t).$$

FIG. 4 Circuit that we will analyze. Initially all voltages are set to 1 and then suddenly the input voltage, V_{in}, is pulled down to zero. A measurement is made at the voltage between the resistor and capacitor, V_{out}.

Since the components are in series, the current through the resistor, I_R, must equal that through the capacitor, I_C. Recall that this statement is invoking Kirchhoff's current law. We therefore have,

$$C\frac{dV_{\text{out}}(t)}{dt} = -\frac{V_{\text{out}}(t)}{R}.$$

The equation can be rewritten as

$$\frac{dV_{\text{out}}(t)}{dt} = -\frac{V_{\text{out}}(t)}{RC}.$$

We can notice from the units here, that the product RC has units of time. If we define the time constant as $\tau = RC$, then you should notice the analogy with the draining tank.

$$\frac{dV_{\text{out}}(t)}{dt} = -\frac{V_{\text{out}}(t)}{\tau}.$$

The equation is exactly the same, only the symbols and physical meaning is completely changed. Just as with the hydraulic example, the time constant $\tau = RC$ increases with an increase in the both resistance and capacity(ance).

Even though we could use the hydraulic analogy to get the solution, let's work through the details again. Separating the variables of this equation,

$$\frac{dV_{\text{out}}}{V_{\text{out}}} = \frac{-dt}{\tau},$$

and integrating yields,

$$\ln(V_{\text{out}}) = \frac{-t}{\tau} + C,$$

where C is a constant of integration. Applying the initial condition $V_{\text{out}}(t = 0) = 1$ yields,

$$\ln(1) = \frac{-0}{\tau} + C \quad \text{or} \quad C = 0.$$

Taking the exponential of both sides of the equation we have

$$V_{\text{out}} = e^{-t/\tau}.$$

The solution to the problem is the same as in the hydraulic analogy. Experimental data are shown in Figure 5.
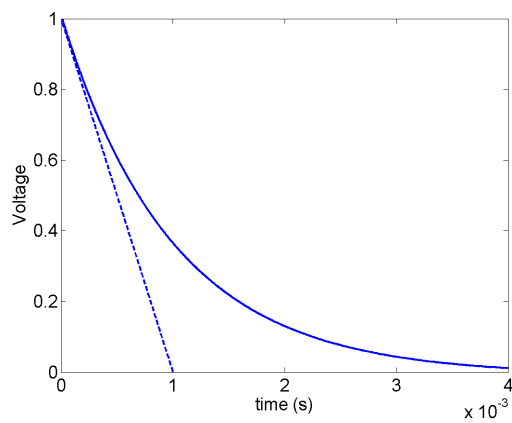
FIG. 5 Measurement of voltage between resistor and capacitor in circuit of Figure 4. The value of the time constant was selected to be $RC = 0.001$ s. The dashed line shows the linear extrapolation using the 1 ms time constant. The measured behavior is extremely close to the predicted behavior.