Introduction to Sensors, Instrumentation, and Measurement

October 31, 2015

1.4 Another Stupid Op-Amp Trick

In this section, we consider another useful op-amp circuit, using the ideas that we developed in the Section 1.1 to analyze and to reason about its behavior. As before, we shall assume that the circuit is made from a rail-to-rail op amp that is powered from a single-ended supply from 0 V to V_{dd} unless otherwise stated.

1.4.1 Band-Pass Amplifier/Filter

Consider the circuit shown in Fig. 10 comprising an op amp, two resistors, R_1 and R_2, and two capacitors, C_1 and C_2. As with the first-order low-pass filter/amplifier circuit, the op amp's output voltage, V_{out}, which is the output of the circuit as a whole, is fed back into the op amp's inverting input through both R_2 and C_2, which are in parallel with each other. As with the first-order high-pass filter/amplifier circuit, this circuit's input voltage, V_{in}, couples into the op amp's inverting input through C_1 and R_1, which are connected in series with each other. The op amp's noninverting input is held at a constant reference voltage, V_{ref}.

By the second observation of Section 1.1, if V_{out} is not to be stuck at one of the rails, then the op amp's inverting voltage, V_2, must be equal to its noninverting input voltage, V_{ref}. So, by Ohm's law, we have that

$$I_1 = \frac{V_1 - V_{\text{ref}}}{R_1} \tag{1}$$

and that

$$I_2 = \frac{V_{\text{ref}} - V_{\text{out}}}{R_2}. \tag{2}$$

Similarly, from the capacitor's current-voltage relationship, we have that

$$I_{\text{c1}} = C_1 \frac{d}{dt}\left(V_{\text{in}} - V_1\right) \tag{3}$$

and that

$$I_{\text{c2}} = C_2 \frac{d}{dt}\left(V_{\text{ref}} - V_{\text{out}}\right). \tag{4}$$

By applying KCL at the node between C_1 and R_1 and by using Eq. 1 and Eq. 3, we have that

$$\underbrace{C_1 \frac{d}{dt}\left(V_{\text{in}} - V_1\right)}_{I_{\text{c1}}} = \underbrace{\frac{V_1 - V_{\text{ref}}}{R_1}}_{I_1},$$

which we can rearrange to find that

$$\underbrace{R_1 C_1}_{\tau_1} \frac{d}{dt}\left(V_{\text{in}} - V_{\text{ref}} + V_{\text{ref}} - V_1\right) = \left(V_1 - V_{\text{ref}}\right),$$

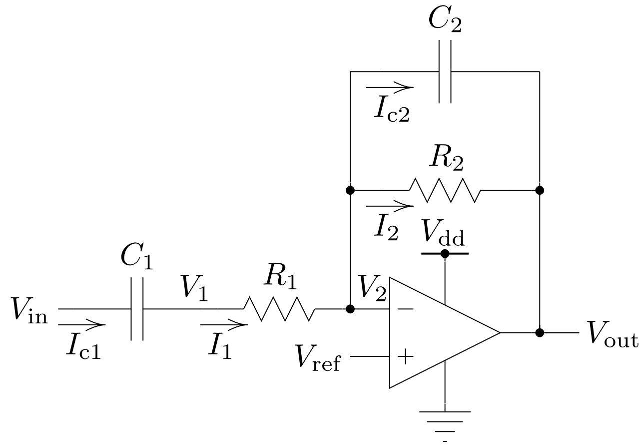

Figure 10: Schematic of a band-pass filter/amplifier circuit. The corner frequency of the high-pass part of circuit's response is given by $\omega_{c1} = 1/\tau_1 = 1/R_1C_1$, and the corner frequency of the low-pass part of the circuit's response is given by $\omega_{c2} = 1/\tau_2 = 1/R_2C_2$. Assuming that $\omega_{c1} \ll \omega_{c2}$, the circuit will have a flat pass band extending from ω_{c1} to ω_{c2} with a pass-band gain of $A = -R_2/R_1$. Outside of the pass band, the input will be attenuated at a rate of 20 dB/decade.

where we have introduced a first time constant, $\tau_1 = R_1C_1$, and added zero in the form of $V_{ref} - V_{ref}$ inside the time derivative on the left side of the equation. We can further rearrange this equation to find that

$$\tau_1 \frac{d}{dt}(V_{in} - V_{ref}) - \tau_1 \frac{d}{dt}(V_1 - V_{ref}) = (V_1 - V_{ref})$$

or equivalently

$$\tau_1 \frac{d}{dt}(V_1 - V_{ref}) + (V_1 - V_{ref}) = \tau_1 \frac{d}{dt}(V_{in} - V_{ref}).$$

By introducing shifted voltage variables, $U_{in} = V_{in} - V_{ref}$ and $U_1 = V_1 - V_{ref}$, into this equation, we have that

$$\tau_1 \frac{dU_1}{dt} + U_1 = \tau_1 \frac{dU_{in}}{dt}. \tag{5}$$

This equation is the governing equation for a first-order high-pass filter with a pass-band gain of unity and a corner frequency at $\omega_{c1} = 1/\tau_1$, so the intermediate node voltage V_1 is a high-pass-filtered version of V_{in}.

By applying KCL to the node connected to the op amp's inverting input and by using Eq. 1, Eq. 2, and Eq. 4, we have that

$$\underbrace{\frac{V_1 - V_{ref}}{R_1}}_{I_1} = \underbrace{\frac{V_{ref} - V_2}{R_2}}_{I_2} + \underbrace{C_2 \frac{d}{dt}(V_{ref} - V_{out})}_{I_{c2}},$$

which we can rearrange by multiplying both sides by $-R_2$ to find that

$$\underbrace{-\frac{R_2}{R_1}}_{A}(V_1 - V_{ref}) = (V_{out} - V_{ref}) + \underbrace{R_2C_2}_{\tau_2} \frac{d}{dt}(V_{out} - V_{ref}),$$

where we have introduced a second time constant, $\tau_2 = R_2C_2$, and a gain, $A = -R_2/R_1$. By introducing shifted voltage variables, $U_1 = V_1 - V_{\rm ref}$ and $U_{\rm out} = V_{\rm out} - V_{\rm ref}$, into this equation, we have that

$$\tau_2 \frac{dU_{\rm out}}{dt} + U_{\rm out} = AU_1. \tag{6}$$

This equation is the governing equation for a first-order low-pass filter/amplifier with a pass-band gain of A and a corner frequency at $\omega_{\rm c2} = 1/\tau_2$. So, the output of this circuit will be an amplified, low-pass-filtered version of V_1, which, in turn, is a high-pass-filtered version of $V_{\rm in}$. Consequently, as a whole, this circuit is a band-pass filter/amplifier. Assuming that we choose R_1, R_2, C_1, and C_2 so that $\omega_{\rm c1} \ll \omega_{\rm c2}$, the circuit will have a flat pass band extending from $\omega_{\rm c1}$ to $\omega_{\rm c2}$ with a pass-band gain of A. Outside of the pass band, the input will be attenuated at a rate of 20 dB/decade (i.e., one decade for each decade of frequency beyond the corners in either direction).

Now, we shall use some of the ideas that we have been developing about complex numbers and complex exponentials to analyze the response of this circuit to a sinusoidal input signal at some angular frequency, ω. To do so, we shall assume that an input signal, $U_{\rm in}$, of the form

$$U_{\rm in}(t) = U_0 e^{j\omega t}, \tag{7}$$

where U_0 is a (real) constant voltage amplitude, gives rise to an intermediate output, U_1, of the form

$$U_1(t) = A_1 U_{\rm in}(t), \tag{8}$$

where A_1 is a (complex) dimensionless gain factor (possibly a function of ω, but not a function of time) that accounts for both changes in amplitude (via the magnitude) and in phase (via the angle) of the intermediate output signal relative to the input signal due to the high-pass filter. We shall also assume that the intermediate output, U_1, gives rise to an output signal, $U_{\rm out}$, of the form

$$U_{\rm out}(t) = A_2 U_1(t) = A_2 A_1 U_{\rm in}(t) = A_2 A_1 U_0 e^{j\omega t}, \tag{9}$$

where A_2 is a (complex) dimensionless gain factor that accounts for both changes in amplitude and in phase of the output signal relative to the intermediate output signal due to the low-pass filter/amplifier.

By substituting Eq. 7 and Eq. 8 into Eq. 5, we find that

$$\tau_1 \frac{d}{dt}\left(A_1 U_0 e^{j\omega t}\right) + A_1 U_0 e^{j\omega t} = \tau_1 \frac{d}{dt}\left(U_0 e^{j\omega t}\right),$$

which implies that

$$j\omega\tau_1 A_1 U_0 e^{j\omega t} + A_1 U_0 e^{j\omega t} = j\omega\tau_1 U_0 e^{j\omega t}.$$

By dividing both sides of this equation by $U_0 e^{j\omega t}$ (which is never zero), we obtain

$$A_1\left(j\omega\tau_1 + 1\right) = j\omega\tau_1,$$

which we can solve for A_1 to find that

$$A_1 = \frac{j\omega\tau_1}{1 + j\omega\tau_1} = \frac{\omega\tau_1 \cdot e^{j\pi/2}}{\sqrt{1 + (\omega\tau_1)^2} \cdot e^{j\tan^{-1}(\omega\tau_1/1)}} = \frac{\omega\tau_1}{\sqrt{1 + (\omega\tau_1)^2}} \cdot e^{j\left((\pi/2) - \tan^{-1}(\omega\tau_1)\right)}.$$

Similarly, by substituting Eq. 8 and Eq. 9 into Eq. 6, we find that

$$\tau_2 \frac{d}{dt}\left(A_2 A_1 U_0 e^{j\omega t}\right) + A_2 A_1 U_0 e^{j\omega t} = A A_1 U_0 e^{j\omega t},$$

which implies that

$$j\omega\tau_2 A_2 A_1 U_0 e^{j\omega t} + A_2 A_1 U_0 e^{j\omega t} = A A_1 U_0 e^{j\omega t}.$$

By dividing both sides of this equation by $A_1 U_0 e^{j\omega t}$ (which is never zero if $\omega \neq 0$), we find that

$$A_2\left(j\omega\tau_2 + 1\right) = A,$$

which we can rearrange to solve for A_2, thereby obtaining

$$A_2 = \frac{A}{1 + j\omega\tau_2} = \frac{|A| e^{j\pi}}{\sqrt{1 + \left(\omega\tau_2\right)^2} \cdot e^{j\tan^{-1}(\omega\tau_2/1)}} = \frac{|A|}{\sqrt{1 + \left(\omega\tau_2\right)^2}} \cdot e^{j\left(\pi - \tan^{-1}(\omega\tau_2)\right)}.$$

From Eq. 9, the overall response of the circuit to a sinusoidal input is characterized by

$$\begin{aligned} A_1 A_2 &= \frac{j\omega\tau_1}{1 + j\omega\tau_1} \cdot \frac{A}{1 + j\omega\tau_2} \\ &= \frac{\omega\tau_1}{\sqrt{1 + \left(\omega\tau_1\right)^2}} \cdot \frac{|A|}{\sqrt{1 + \left(\omega\tau_2\right)^2}} \cdot e^{j\left((3\pi/2) - \tan^{-1}(\omega\tau_1) - \tan^{-1}(\omega\tau_2)\right)}, \end{aligned}$$

where $A = -R_2/R_1$, $\tau_1 = R_1 C_1$, and $\tau_2 = R_2 C_2$.