

Analysis of RC filters

Low-pass filter

Consider a resistor and capacitor arranged in series as shown in Figure 1. This circuit is the same one we studied last week. In last week's lab we drove this circuit with a low frequency square wave at V_{in} and watched the voltage at V_{out} exponentially converge as the capacitor charged. We also saw that if we increased the frequency of the square wave, we reached a state where the capacitor didn't fully charge such that V_{out} was less than V_{in} . As we continued to increase the frequency, the capacitor had no time to charge and thus the voltage V_{out} became relatively constant at the the average value of V_{in} .

Now let's consider this same problem a little more analytically but using a sinusoidal input voltage. The resistor is driven with an external sources as $V_{in} = V \sin(\omega t)$ where the frequency ω would be in radians/second and is something we control. Since the circuit is composed of *linear* components, then the output voltage must of the same frequency as the input signal. By linear components we mean that the voltage and current are linearly related - not that the current is proportional to the square of the voltage or some other such law. At this point, we will ask you to accept that the frequency of the output doesn't change and we can check this experimentally in the lab to confirm the assumption. While the frequency of the output can't change, the amplitude can and there can also be a shift in phase from output to input. For example, if the input was a sine and the output was a cosine, we would say the output shifted in phase by 90 degrees. In general, the output will have a form $V_{out} = AV \sin(\omega t + \phi)$. Here, A represents the relative amplitude of the output compared to the input and ϕ the phase.

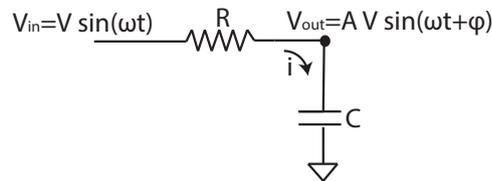


FIG. 1 Low-pass filter circuit with resistor and capacitor in series.

Now let's use our basic circuit laws to analyze this circuit in more detail. The current through the resistor is,

$$I = \frac{V_{in} - V_{out}}{R}.$$

and the current through the capacitor is,

$$I = C \frac{dV_{out}}{dt}.$$

Since the two parts are in series, Kirchoff's current law would state that the current through the resistor must equal that through the capacitor,

$$RC \frac{dV_{out}}{dt} = V_{in} - V_{out}.$$

At this point the resulting equation is exactly what we saw last week.

Experimentally, if we check the output of this circuit we would see that while perhaps there is some short transient behavior, very quickly the output reaches a sinusoidal steady state. The output of the circuit is a sine wave of a certain amplitude and a phase relative to the input sine wave. This output sine wave repeats and repeats with no change. We are interested in figuring out the amplitude and phase once the output has reached this periodic steady state.

If we assume the generic sinusoidal forms of the input and output voltage and substitute those into our equations, $V_{in} = V \sin(\omega t)$ and $V_{out} = AV \sin(\omega t + \phi)$, we have

$$RC\omega AV \cos(\omega t + \phi) = V \sin(\omega t) - AV \sin(\omega t + \phi).$$

Using trigonometric identities (that I always forget but can easily look up on wikipedia), we can obtain

$$RC\omega A (\cos(\omega t)\cos(\phi) - \sin(\omega t)\sin(\phi)) = \sin(\omega t) - A (\sin(\omega t)\cos(\phi) + \cos(\omega t)\sin(\phi)).$$

The only way this equation can be true at all times is that if all the terms with $\cos(\omega t)$ and $\sin(\omega t)$ balance separately. Grouping all the terms with cosine together we have

$$RC\omega A \cos(\omega t)\cos(\phi) = -A \cos(\omega t)\sin(\phi)$$

Canceling $A \cos(\omega t)$ from both sides gives,

$$RC\omega \cos(\phi) = -\sin(\phi)$$

or

$$\tan(\phi) = -RC\omega.$$

This equation tells us the phase between the output and the input. There are a few interesting things about this equation. First, we notice that the product RC must have units of time (seconds if we express R in ohms and C in farads). Since the tangent is a unitless function, the term $RC\omega$ must have no units. Second, we can also easily see the low and high frequency limits - when $\omega \rightarrow 0$, then $\phi \rightarrow 0$ and when $\omega \rightarrow \infty$, then $\phi \rightarrow -90$ degrees. Finally, when $RC\omega = 1$ then we are halfway between the low and high frequency limit with $\phi = 45$ degrees. The frequency given by $\frac{1}{RC}$ radians/sec provides the measure of what we consider to be “high” or “low” frequency.

Now we go back and group the $\sin(\omega t)$ terms to obtain,

$$-RC\omega A \sin(\omega t)\sin(\phi) = -A \sin(\omega t)\cos(\phi) + \sin(\omega t).$$

Canceling the \sin term and rearranging we obtain,

$$A = \frac{1}{\cos(\phi) - RC\omega \sin(\phi)}.$$

This expression gives the amplitude of the output divided by the amplitude of the input. In the low frequency limit, we have $A \rightarrow 1$ and at high frequency we have $A \rightarrow \frac{1}{RC\omega}$.

We can get a better handle on what is happening by generating a plot. We can easily generate this plot in MATLAB as follows.

- `RC = 1;`
- `omega = logspace(-2,2,100);`
- `phi = atan(-RC*omega);`
- `A = 1./(cos(phi) - RC*omega.*sin(phi));`
- `subplot(2,1,1); loglog(omega,A);`
- `subplot(2,1,2); semilogx(omega,phi*90/(pi/2));`

The result of these commands is shown in Figure 2.

This plot tells us everything we ever would hope to know about this circuit. It tells us how the output amplitude and phase change as a function of frequency. This type of plot is usually referred to as a Bode (Bo Dee) plot. This circuit is called a low-pass filter as it tends to allow low frequency signals through unchanged. At low frequency the output amplitude is the same as the input and the phase between output and input is zero. At high frequency the output amplitude decreases as the frequency increases. Notice that by convention, I have used log-log and linear-log coordinates for the amplitude and phase. The use of log coordinates on the x-axis allows us to more clearly see what is happening over many orders of magnitude in frequency (4 in this case). The use of log coordinates on the y-axis in the amplitude plot allows us to see how the signal decreases over many orders of magnitude. In this case we can clearly see from the plot, that at high frequency, a factor of ten increase in frequency gives a factor of 10 decrease in amplitude. Power law functions show up as straight lines on log-log plots.

A better form for our answer uses the trigonometric identities to reduce the amplitude equation to a more explicit form. We are doing nothing more than re-arranging the equations into a simpler form. Namely we can use the relations that state,

$$\cos(-\tan^{-1}(RC\omega)) = \frac{1}{\sqrt{1 + R^2 C^2 \omega^2}},$$

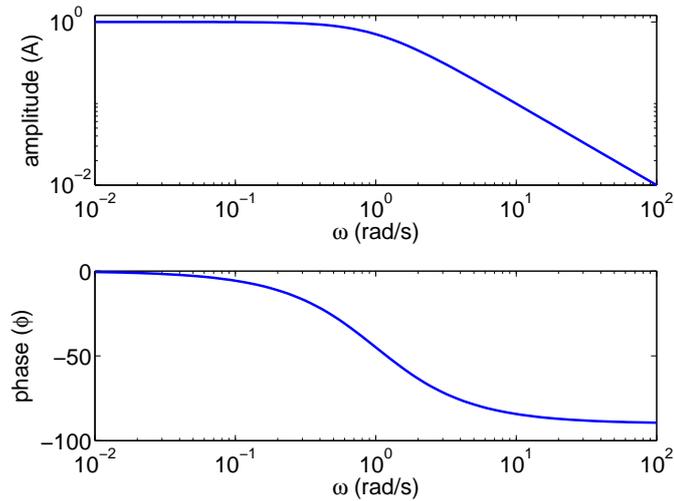


FIG. 2 Low pass filter. Relative amplitude as a function of frequency and the phase angle in degrees as a function of frequency when $RC = 1$.

$$\sin(-\tan^{-1}(RC\omega)) = \frac{-RC\omega}{\sqrt{1 + R^2C^2\omega^2}}.$$

Again, these are trigonometric identities that I would not expect you remember but are easy to look up.

Substituting these trig identities into the amplitude equation provides,

$$A = \frac{1}{\frac{1}{\sqrt{1+R^2C^2\omega^2}} + \frac{R^2C^2\omega^2}{\sqrt{1+R^2C^2\omega^2}}} = \frac{1}{\sqrt{1 + R^2C^2\omega^2}}$$

This form of the amplitude equation provides a much simpler interpretation of the Bode plot than the implicit amplitude equation in terms of the trigonometric functions. When $RC\omega \rightarrow 0$ then $A \rightarrow 1$. When $RC\omega \rightarrow \infty$ then $A \rightarrow \frac{1}{RC\omega}$.

High-pass filter

Now let's change the order of the parts and see what happens. The current through the resistor is

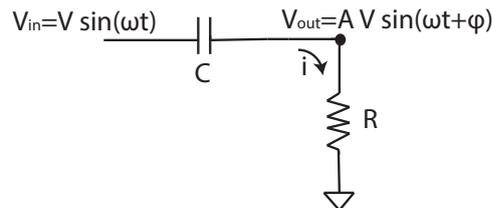


FIG. 3 High-pass filter circuit.

$$I = \frac{V_{\text{out}}}{R}.$$

and the current through the capacitor is,

$$I = C \frac{d(V_{\text{in}} - V_{\text{out}})}{dt}.$$

Thus

$$RC \frac{d(V_{\text{in}} - V_{\text{out}})}{dt} = V_{\text{out}}.$$

Substituting in the generic sinusoidal form as before gives us,

$$RC\omega (V \cos(\omega t) - AV \cos(\omega t) \cos(\phi) + AV \sin(\omega t) \sin(\phi)) = AV (\sin(\omega t) \cos(\phi) + \cos(\omega t) \sin(\phi))$$

Grouping the sine terms together gives,

$$\tan(\phi) = \frac{1}{RC\omega}.$$

and the cosine terms give,

$$A = \frac{RC\omega}{\sin(\phi) + RC\omega \cos(\phi)}.$$

At this point we could use MATLAB to create the plot of amplitude and phase as a function of frequency, simply modifying the script from above. The plot of the amplitude and phase as a function of frequency is shown in Figure 4.

We could also rearrange the equation and get a different form by using our trig identities again,

$$\cos\left(\tan^{-1}\left(\frac{1}{RC\omega}\right)\right) = \frac{1}{\sqrt{1 + \frac{1}{R^2C^2\omega^2}}} = \frac{RC\omega}{\sqrt{R^2C^2\omega^2 + 1}},$$

$$\sin\left(\tan^{-1}\left(\frac{1}{RC\omega}\right)\right) = \frac{\frac{1}{RC\omega}}{\sqrt{1 + \frac{1}{R^2C^2\omega^2}}} = \frac{1}{\sqrt{R^2C^2\omega^2 + 1}},$$

Combining the expressions gives the amplitude as

$$A = \frac{RC\omega}{\sqrt{R^2C^2\omega^2 + 1}}.$$

Here the behavior is opposite as before. As the frequency goes to infinity the phase goes to zero and the relative amplitude, A , becomes 1. As the frequency goes to zero, the phase goes to 90 degrees and the relative amplitude becomes $A \rightarrow RC\omega$.

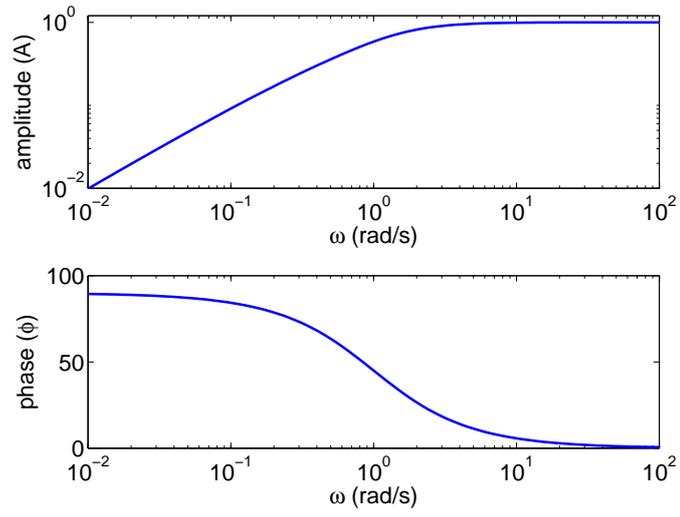


FIG. 4 High pass filter. Relative amplitude as a function of frequency and the phase angle in degrees as a function of frequency when $RC = 1$.