Introduction to Sensors, Instrumentation, and Measurement

November 12, 2015

1.5 Gain/Bandwidth Trade-Off

In this section, we shall revisit the second of our observations about op amps of Section 1.1, which we have used extensively in our analysis of op-amp circuits so far. Recall that it was based on examining the op amp's VTC, which is a plot of a circuit's steady-state output voltage as a function of its input voltage. By using this observation, we are making the assumption that the system is operating *quasistatically*—that is, we are assuming that all of the voltages and currents in the circuit are varying on a much slower timescale than that which characterizes the op amp's internal dynamics, which is typified by the op amp's gain-bandwidth product,[1] ω_1. In order to account for the op amp's internal dynamics in analyzing a given circuit, we shall replace our assumption that, in order for the op amp's output not to be stuck at one of the rails, the two input voltages must be equal to each other with the assumption that the op amp's output voltage changes at a rate that is given by

$$\frac{dV_{\mathrm{out}}}{dt} = \omega_1 \left(V_{\mathrm{pos}} - V_{\mathrm{neg}}\right). \tag{1}$$

However, for op-amp circuits operating on a single-ended supply where we typically reference our voltages to a reference voltage, V_{ref}, we shall often find it more convenient to add zero to both sides of this equation to obtain

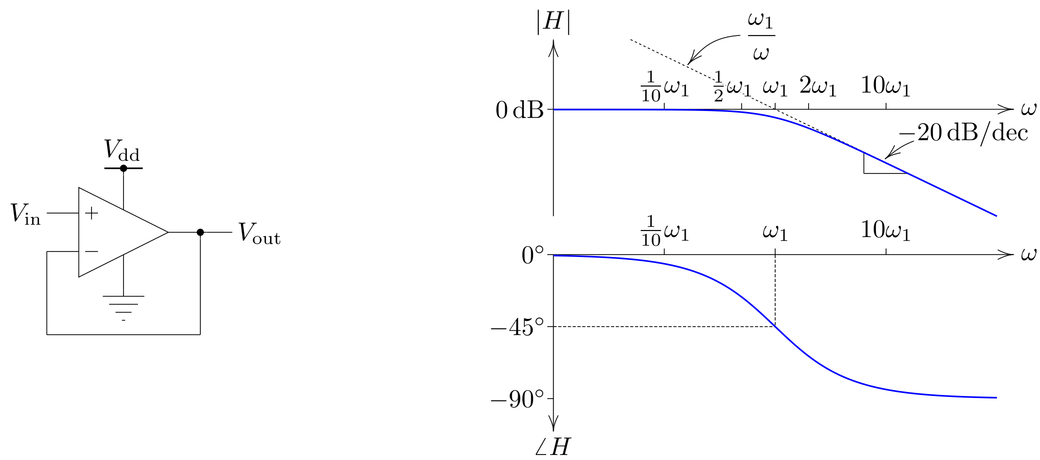
$$\frac{dV_{\mathrm{out}}}{dt} - \frac{dV_{\mathrm{ref}}}{dt} = \omega_1 \left(V_{\mathrm{pos}} - V_{\mathrm{ref}} + V_{\mathrm{ref}} - V_{\mathrm{neg}}\right),$$

or equivalently

$$\frac{d}{dt}\left(V_{\mathrm{out}} - V_{\mathrm{ref}}\right) = \omega_1 \left(\left(V_{\mathrm{pos}} - V_{\mathrm{ref}}\right) - \left(V_{\mathrm{neg}} - V_{\mathrm{ref}}\right)\right). \tag{2}$$

In order to see the effects of the op amp's internal dynamics, we shall use Eq. 1 or Eq. 2 to analyze the unity-gain follower, the noninverting amplifier, and the inverting amplifier. In each case, we shall find the circuit is actually a low-pass filter/amplifier, behaving as we had previously analyzed only at low frequencies. We shall also see that there is a trade-off between its (low-frequency) gain and its bandwidth so that the product of an amplifier's gain and its bandwidth is nearly equal to the op amp's gain-bandwidth product (whence the name). For example, if our op amp has a gain-bandwidth product of 1 MHz and we use it to make a unity-gain follower, the follower will have a bandwidth of 1 MHz. If we use that same op amp to make an amplifier with a gain of 10, the amplifier will have a bandwidth of 1 MHz/10 = 100 kHz. If we use it to make an amplifier with a gain of 100, the resulting amplifier will have a bandwidth of 1 MHz/100 = 10 kHz. If we use it to make an amplifier with a gain of 1000, the resulting amplifier would have a bandwidth of 1 MHz/1000 = 1 kHz, and so on.

[1] This assumption holds so long as the op amp's input voltage difference is small enough that the rate of change of the op amp's output voltage is not limited by the op amp's slew rate.



(a) (b)

Figure 11: Unity-gain follower (a) circuit schematic and (b) frequency response characteristic (i.e., magnitude and phase Bode plots). This circuit behaves as a low-pass filter with a pass-band gain of unity and a corner frequency at ω_1. It acts as we found in Section 1.2.1 only for low frequencies. Deviations from the ideal behavior start to become discernible in the phase response about an order of magnitude below the corner frequency (i.e., at $\omega \approx \frac{1}{10}\omega_1$) and in the magnitude at about an octave below the corner frequency (i.e., at $\omega \approx \frac{1}{2}\omega_1$).

1.5.1 Unity-Gain Follower/Buffer Revisited

Consider the unity-gain follower circuit, shown in Fig. 11a. To investigate how the op amp's internal dynamics shape the behavior of this circuit at frequencies approaching the op amp's gain-bandwidth product, we make use of Eq. 1 in place of the second observation of Section 1.1, which for this circuit implies that

$$\frac{dV_{\text{out}}}{dt} = \omega_1 \left(V_{\text{in}} - V_{\text{out}}\right),$$

which we can rearrange to find that

$$\frac{1}{\omega_1}\frac{dV_{\text{out}}}{dt} + V_{\text{out}} = V_{\text{in}}. \tag{3}$$

This equation is the governing equation for a first-order low-pass filter with a pass-band gain of unity and a corner frequency at ω_1.

To analyze the response of the unity-gain follower to a sinusoidal input signal at some angular frequency, ω, we assume that an input signal of the form

$$V_{\text{in}}(t) = V_0 e^{j\omega t}, \tag{4}$$

where V_0 is a (real) constant voltage amplitude, gives rise to an output signal of the form

$$V_{\text{out}}(t) = H V_0 e^{j\omega t}, \tag{5}$$

where H is a (complex) dimensionless gain factor (possibly a function of ω, but not a function of time) that accounts for both changes in amplitude (via the magnitude) and in phase (via the angle) of the output signal relative to the input signal.

By substituting Eq. 4 and Eq. 5 into Eq. 3, we find that

$$\frac{1}{\omega_1}\frac{d}{dt}\left(HV_0e^{j\omega t}\right) + HV_0e^{j\omega t} = V_0e^{j\omega t},$$

which implies that

$$\frac{j\omega}{\omega_1}HV_0e^{j\omega t} + HV_0e^{j\omega t} = V_0e^{j\omega t}.$$

By dividing both sides of this equation by $V_0e^{j\omega t}$, we obtain

$$H\left(\frac{j\omega}{\omega_1} + 1\right) = 1,$$

which we can solve for H to find that

$$H = \frac{1}{1 + j\omega/\omega_1} = \frac{1}{\sqrt{1 + (\omega/\omega_1)^2}} \cdot e^{-j\tan^{-1}(\omega/\omega_1)}.$$

Fig. 11b shows magnitude and phase Bode plots of H as a function of ω for the unity-gain follower. On this analysis, the circuit is a low-pass filter with a corner frequency at ω_1, behaving as we found in Section 1.2.1 only at low frequencies. Deviations from the ideal behavior start to become noticeable in the phase response about an order of magnitude below the corner frequency (i.e., at $\omega \approx \frac{1}{10}\omega_1$) and in the magnitude at about an octave below the corner frequency (i.e., at $\omega \approx \frac{1}{2}\omega_1$).

1.5.2 Noninverting Amplifier Revisited

Consider the noninverting amplifier circuit, shown in Fig. 12a. In order to investigate how the op amp's dynamics affects the response of this circuit as the input signal frequency approaches the op amp's gain-bandwidth product, we shall make use of Eq. 2 where we had previously used the second observation of Section 1.1. Applying this equation to the circuit of Fig. 12a, we have that

$$\frac{d}{dt}\left(V_{\text{out}} - V_{\text{ref}}\right) = \omega_1\left(\left(V_{\text{in}} - V_{\text{ref}}\right) - \left(V - V_{\text{ref}}\right)\right). \quad (6)$$

Because op amp's inverting input draws no current, then the voltage across R_2, $V - V_{\text{ref}}$, is related to the total voltage across the series combination of R_1 and R_2, $V_{\text{out}} - V_{\text{ref}}$, by through a resistive voltage divider ratio, given by

$$V - V_{\text{ref}} = \left(V_{\text{out}} - V_{\text{ref}}\right)\frac{R_2}{R_1 + R_2}. \quad (7)$$

By substituting Eq. 7 into Eq. 6, we obtain

$$\frac{d}{dt}\left(V_{\text{out}} - V_{\text{ref}}\right) = \omega_1\left(\left(V_{\text{in}} - V_{\text{ref}}\right) - \left(V_{\text{out}} - V_{\text{ref}}\right)\frac{R_2}{R_1 + R_2}\right),$$

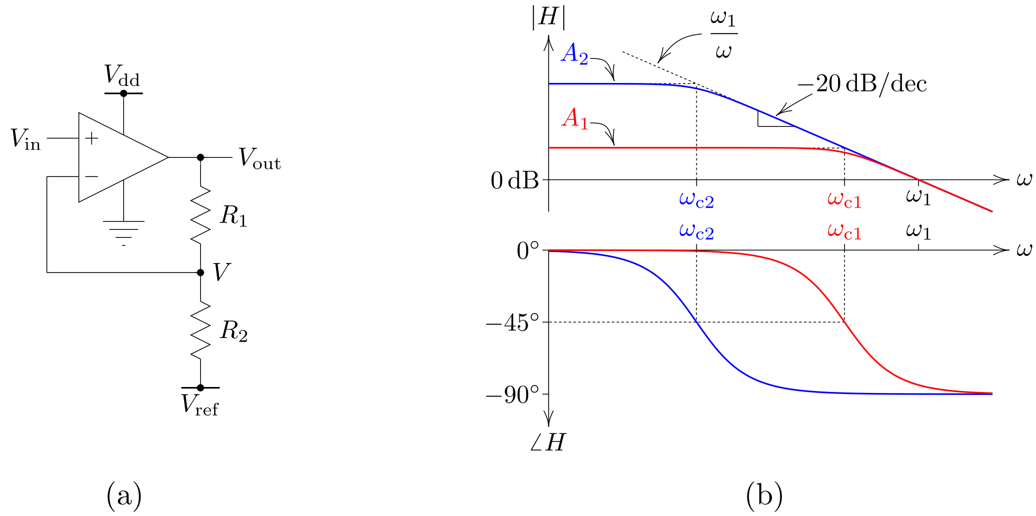


Figure 12: Noninverting amplifier (a) circuit schematic and (b) frequency response characteristic (i.e., magnitude and phase Bode plots) for two values of the low-frequency gain: A_1 (red) and A_2 (blue). This circuit behaves as a low-pass filter with a pass-band gain of $A = 1 + R_1/R_2$ and a corner frequency at $\omega_c = \omega_1/A$. So, it acts as we found in Section 1.2.2 only for low frequencies (i.e., $\omega \ll \omega_c$). Note that A and ω_c are inversely related to each other so that there is an inherent trade-off between gain and bandwidth for this circuit.

which we can rearrange to find that

$$\frac{1}{\omega_1}\frac{d}{dt}\left(V_{\rm out} - V_{\rm ref}\right) + \frac{R_2}{R_1 + R_2}\left(V_{\rm out} - V_{\rm ref}\right) = \left(V_{\rm in} - V_{\rm ref}\right).$$

By multiplying both sides of this equation by $1 + R_1/R_2$, we have that

$$\overbrace{\Bigg(\underbrace{1 + \frac{R_1}{R_2}}_{A}\Bigg)\frac{1}{\omega_1}}^{1/\omega_c}\frac{d}{dt}\left(V_{\rm out} - V_{\rm ref}\right) + \left(V_{\rm out} - V_{\rm ref}\right) = \Bigg(\underbrace{1 + \frac{R_1}{R_2}}_{A}\Bigg)\left(V_{\rm in} - V_{\rm ref}\right),$$

where $A = 1 + R_1/R_2$ is the (low-frequency) gain of the noninverting amplifier and $\omega_c = \omega_1/A$ is a corner frequency in the noninverting amplifier's response due to the op amp's internal dynamics. By defining shifted voltage variables by $U_{\rm in} = V_{\rm in} - V_{\rm ref}$ and $U_{\rm out} = V_{\rm out} - V_{\rm ref}$, we can re-express this equation as

$$\frac{1}{\omega_c}\frac{dU_{\rm out}}{dt} + U_{\rm out} = AU_{\rm in}, \tag{8}$$

which is the governing equation for a first-order low-pass filter/amplifier with a pass-band gain of A and a corner frequency at ω_c.

To analyze the response of the noninverting amplifier to a sinusoidal input signal at some angular frequency, ω, we assume that an input signal of the form

$$U_{\rm in} = U_0 e^{j\omega t}, \tag{9}$$

where U_0 is a (real) constant voltage amplitude, results in an output signal of the form

$$U_{\text{out}} = HU_0 e^{j\omega t}, \tag{10}$$

where H is a (complex) dimensionless gain factor (possibly a function of ω, but not a function of time) that accounts for both changes in amplitude (via the magnitude) and in phase (via the angle) of the output signal relative to the input signal.

By substituting Eq. 9 and Eq. 10 into Eq. 8, we find that

$$\frac{1}{\omega_{\text{c}}}\frac{d}{dt}\left(HU_0 e^{j\omega t}\right) + HU_0 e^{j\omega t} = AU_0 e^{j\omega t},$$

which implies that

$$\frac{j\omega}{\omega_{\text{c}}} HU_0 e^{j\omega t} + HU_0 e^{j\omega t} = AU_0 e^{j\omega t}.$$

By dividing both sides of this equation by $U_0 e^{j\omega t}$, we have that

$$H\left(\frac{j\omega}{\omega_{\text{c}}} + 1\right) = A,$$

which we can solve for H to obtain

$$H = \frac{A}{1 + j\omega/\omega_{\text{c}}} = \frac{A}{\sqrt{1 + (\omega/\omega_{\text{c}})^2}} \cdot e^{-j\tan^{-1}(\omega/\omega_{\text{c}})}.$$

Fig. 12b shows magnitude and phase Bode plots of H as a function of ω for the noninverting amplifier for two different values of the low-frequency gain: A_1 (red) and A_2 (blue). Note here that the amplifier's corner frequency, ω_{c}, is related to the low-frequency gain in such a way that the higher the gain, the smaller the bandwidth. For this circuit, the two are, in fact, inversely related to each other so that

$$\omega_{\text{c}} = \frac{\omega_1}{A},$$

or equivalently

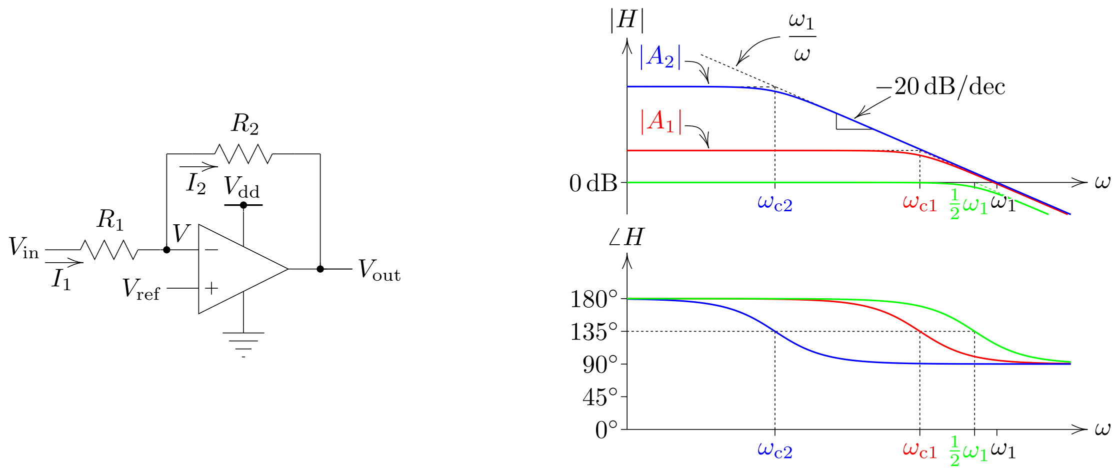
$$A\omega_{\text{c}} = \omega_1,$$

which is why ω_1 is called the *gain-bandwidth product*.

1.5.3 Inverting Amplifier Revisited

Consider the noninverting amplifier circuit, shown in Fig. 13a. In order to investigate how the op amp's dynamics affects the response of this circuit as the input signal frequency approaches the op amp's gain-bandwidth product, we shall make use of Eq. 2 where we had previously used the second observation of Section 1.1. Applying this equation to the circuit of Fig. 13a, we have that

$$\frac{d}{dt}\left(V_{\text{out}} - V_{\text{ref}}\right) = \omega_1\left(\left(V_{\text{ref}} - V_{\text{ref}}\right) - \left(V - V_{\text{ref}}\right)\right) = \omega_1\left(V_{\text{ref}} - V\right),$$



(a) (b)

Figure 13: Inverting amplifier (a) circuit schematic and (b) frequency response characteristic (i.e., magnitude and phase Bode plots) for three values of the low-frequency gain: minus one (green), A_1 (red), and A_2 (blue). This circuit behaves as a low-pass filter/amplifier with a pass-band gain of $A = -R_2/R_1$ and a corner frequency at $\omega_{\text{c}} = \omega_1/(1-A)$. So, it acts as we found in Section 1.2.3 only for low frequencies (i.e., $\omega \ll \omega_{\text{c}}$). Note that, for $|A| \gg 1$, $|A|$ and ω_{c} are inversely related to each other so that there is an inherent trade-off between gain and bandwidth for this circuit.

which we can solve for V to obtain

$$V = V_{\text{ref}} - \frac{1}{\omega_1}\frac{d}{dt}\left(V_{\text{out}} - V_{\text{ref}}\right). \tag{11}$$

Because the op amp's inverting input draws a negligible amount of current, by applying KCL at node V and Ohm's law for R_1 and R_2, we find that

$$\underbrace{\frac{V_{\text{in}} - V}{R_1}}_{I_1} = \underbrace{\frac{V - V_{\text{out}}}{R_2}}_{I_2},$$

which upon multiplying both sides by $-R_2$ becomes

$$V_{\text{out}} - V = -\frac{R_2}{R_1}\left(V_{\text{in}} - V\right).$$

By substituting Eq. 11 into this equation, we find that

$$\begin{aligned}\left(V_{\text{out}} - V_{\text{ref}}\right) + \frac{1}{\omega_1}\frac{d}{dt}\left(V_{\text{out}} - V_{\text{ref}}\right) &= -\frac{R_2}{R_1}\left(V_{\text{in}} - V_{\text{ref}} + \frac{1}{\omega_1}\frac{d}{dt}\left(V_{\text{out}} - V_{\text{ref}}\right)\right) \\ &= -\frac{R_2}{R_1}\left(V_{\text{in}} - V_{\text{ref}}\right) - \frac{R_2}{R_1}\frac{1}{\omega_1}\frac{d}{dt}\left(V_{\text{out}} - V_{\text{ref}}\right),\end{aligned}$$

which we can rearrange to get

$$\overbrace{\underbrace{\left(1+\frac{R_2}{R_1}\right)}_{1-A}\frac{1}{\omega_1}}^{1/\omega_c}\frac{d}{dt}\left(V_{\text{out}}-V_{\text{ref}}\right)+\left(V_{\text{out}}-V_{\text{ref}}\right)=\underbrace{-\frac{R_2}{R_1}}_{A}\left(V_{\text{in}}-V_{\text{ref}}\right),$$

where $A = -R_2/R_1$ is the (low-frequency) gain of the inverting amplifier and $\omega_c = \omega_1/(1-A)$ is a corner frequency in the inverting amplifier's response due to the op amp's internal dynamics. By defining shifted voltage variables $U_{\text{in}} = V_{\text{in}} - V_{\text{ref}}$ and $U_{\text{out}} = V_{\text{out}} - V_{\text{ref}}$, we can re-express this equation as

$$\frac{1}{\omega_c}\frac{dU_{\text{out}}}{dt} + U_{\text{out}} = AU_{\text{in}}, \tag{12}$$

which is the governing equation for a first-order low-pass filter/amplifier with a pass-band gain of A and a corner frequency at ω_c.

To analyze the response of the inverting amplifier to a sinusoidal input signal at some angular frequency, ω, we assume that an input signal of the form

$$U_{\text{in}} = U_0 e^{j\omega t}, \tag{13}$$

where U_0 is a (real) constant voltage amplitude, results in an output signal of the form

$$U_{\text{out}} = HU_0 e^{j\omega t}, \tag{14}$$

where H is a (complex) dimensionless gain factor (possibly a function of ω, but not a function of time) that accounts for both changes in amplitude (via the magnitude) and in phase (via the angle) of the output signal relative to the input signal.

By substituting Eq. 13 and Eq. 14 into Eq. 12, we find that

$$\frac{1}{\omega_c}\frac{d}{dt}\left(HU_0e^{j\omega t}\right) + HU_0e^{j\omega t} = AU_0e^{j\omega t},$$

which implies that

$$\frac{j\omega}{\omega_c}HU_0e^{j\omega t} + HU_0e^{j\omega t} = AU_0e^{j\omega t}.$$

By dividing both sides of this equation by $U_0e^{j\omega t}$, we have that

$$H\left(\frac{j\omega}{\omega_c}+1\right) = A,$$

which we can solve for H to obtain

$$H = \frac{A}{1+j\omega/\omega_c} = \frac{|A|}{\sqrt{1+(\omega/\omega_c)^2}} \cdot e^{j\left(\pi-\tan^{-1}(\omega/\omega_c)\right)}.$$

Fig. 13b shows magnitude and phase Bode plots of H as a function of ω for the inverting amplifier for three different values of the low-frequency gain: minus one (green), A_1 (red), and A_2 (blue). Note here that, as with the noninverting amplifier, the inverting amplifier's corner frequency, ω_c, is related to the low-frequency gain in such a way that the higher the gain, the smaller the bandwidth. For this circuit, the trade-off between the magnitude of its low-frequency gain and its bandwidth is given by

$$|A|\omega_\mathrm{c} = |A|\frac{\omega_1}{1 - A} = \frac{|A|}{1 - A}\omega_1 \approx \omega_1,$$

if $|A| \gg 1$.