INTRODUCTION TO CIRCUIT ANALYSIS

TIMOTHY N. TRICK
University of Illinois
Urbana, Illinois

JOHN WILEY & SONS
New York Santa Barbara Chichester Brisbane Toronto

Library of Congress Cataloging in Publication Data:

Trick, Timothy N 1939–
Introduction to circuit analysis.

Includes bibliographical references and index.
1. Electric circuits. 2. Electric network analysis.
I. Title.
TK454.T74 621.319'2 77-10843
ISBN 0-471-88850-8

Printed in the United States of America

10 9 8 7 6 5 4 3 2

Appendix B
Complex Numbers

The complex number system evolved from the need to take roots of negative real numbers. For example, if $a^2 = -5$, then

$$a = \sqrt{-5} = \sqrt{-1} \cdot \sqrt{5} = j\sqrt{5}$$

a real number preceded by $j = \sqrt{-1}$ is called an imaginary number.[1] A number z of the form

$$z = x + jy \tag{1}$$

is called a complex number, and x is referred to as the *real part* of z, $x = \mathrm{Re}[z]$, and y is called the *imaginary part* of z, $y = \mathrm{Im}[z]$. As in the real number system, certain rules of operation are defined such as addition, multiplication, division, and so on. These rules follow along with a geometric interpretation of z.

B.1 COMPLEX PLANE

A complex number is represented geometrically as illustrated in Figure B.1. The x and y represent coordinates of a point in a set of rectangular cartesian coordinates, and to each point (x, y) there corresponds one and only one complex number (vector) z. Note that we can also specify z uniquely by the angle θ and the magnitude or length of the vector z, that is,

$$z = |z|\underline{/\theta} \tag{2}$$

[1] In mathematical literature the symbol i is often used to denote $\sqrt{-1}$, but we use j to avoid confusion since the symbol i denotes current in electrical circuits.

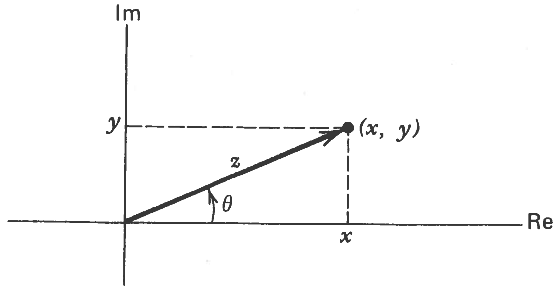


Figure B.1. Complex plane.

where $|z| = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1} y/x$. Equation 2 is referred to as the polar form of z. Note that

$$x = |z| \cos \theta \tag{3}$$

and

$$y = |z| \sin \theta \tag{4}$$

Therefore,

$$z = |z|(\cos \theta + j \sin \theta) \tag{5}$$

Next we show that the complex number $\cos \theta + j \sin \theta$ has an interesting exponential representation.

B.2 THE EXPONENTIAL FORM

In the complex plane the x coordinate of the complex number $\cos \theta + j \sin \theta$ is $\cos \theta$, and its y coordinate is $\sin \theta$. Therefore, since

$$|\cos \theta + j \sin \theta| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1 \tag{6}$$

it follows that this complex number has a magnitude of unity regardless of the angle θ. In addition we note that if

$$z = \cos \theta + j \sin \theta$$

then

$$\frac{dz}{d\theta} = -\sin \theta + j \cos \theta = j(\cos \theta + j \sin \theta)$$

We use the fact that $j \cdot j = \sqrt{-1} \cdot \sqrt{-1} = -1$. Thus,

$$\frac{dz}{d\theta} = jz \tag{7}$$

the solution of this equation is

$$\frac{d}{d\theta} \ln z = j$$

and

$$\ln z = j\theta + c$$

Therefore,

$$z = k\, e^{j\theta} \tag{8}$$

where $k = e^c$. Since

$$z(0) = \cos 0 = 1$$

we conclude that $k = 1$ and

$$z = e^{j\theta} = \cos\theta + j\sin\theta \tag{9}$$

Equation 9 is called the *Euler formula.*

With this background the rules of addition, multiplication, and so on, for complex numbers follow.

B.3 ADDITION OF COMPLEX NUMBERS

Given two complex numbers

$$z_1 = a + jb \tag{10}$$

and

$$z_2 = c + jd \tag{11}$$

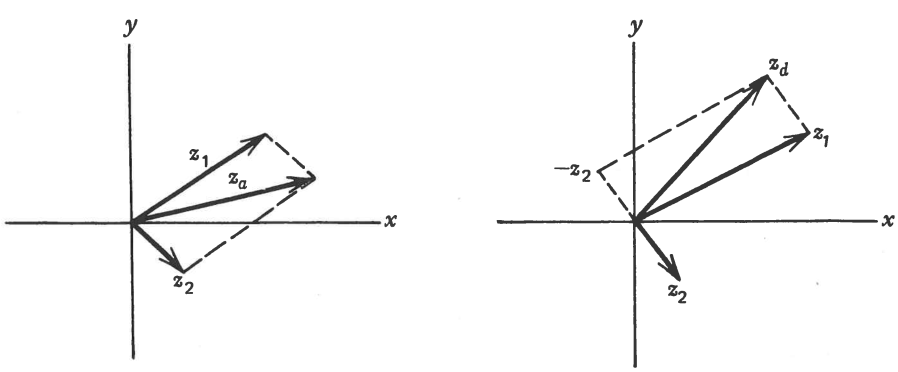
we define their sum as

$$z_a = z_1 + z_2 = (a + c) + j(b + d) \tag{12}$$

and we define their difference as

$$z_d = z_1 - z_2 = (a - c) + j(b - d) \tag{13}$$

These operations result in another complex number. The first closed parenthesis contains the real part of the resultant complex number, and the second closed parenthesis, prefixed by j, contains the imaginary part. The operations are illustrated graphically in Figure B.2.



(a) (b)

Figure B.2. Addition and substraction of complex numbers. (*a*) Addition. (*b*) Subtraction.

B.4 COMPLEX CONJUGATE OF A COMPLEX NUMBER

The complex conjugate of a complex number $z = x + jy$ is defined as

$$z^* = x - jy \tag{14}$$

Note that the imaginary part is multiplied by -1.

B.5 MULTIPLICATION AND DIVISION OF COMPLEX NUMBERS

In polar form the *product* of $z_1 = |z_1|\, e^{j\theta_1}$ with $z_2 = |z_2|\, e^{j\theta_2}$ is defined as

$$z_1 z_2 = |z_1||z_2|\, e^{j(\theta_1+\theta_2)} \tag{15}$$

that is, the new vector has a magnitude equal to the product of the magnitudes of z_1 and z_2 and its angle is the sum of the angles of z_1 and z_2.

In rectangular coordinates this is equivalent to

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1) \tag{16}$$

The *division* of two complex numbers in polar form is defined as

$$\frac{z_1}{z_2} = \frac{|z_1|\, e^{j\theta_1}}{|z_2|\, e^{j\theta_2}} = \frac{|z_1|}{|z_2|}\, e^{j(\theta_1-\theta_2)} \tag{17}$$

Note that the magnitude of divisor is divided into the magnitude of the dividend resulting in the magnitude of the quotient. The angle of the quotient is the difference between the angle of the dividend and the angle of the divisor.

In rectangular coordinates this is equivalent to

$$\frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2} \tag{18}$$

If we multiply the numerator and denominator of (18) by z_2^* we obtain

$$\frac{z_1}{z_2} = \frac{(x_1 x_2 + y_1 y_2) + j(y_1 x_2 - y_2 x_1)}{x_2^2 + y_2^2} \tag{19}$$

It should be clear that these operations are easier to perform in the polar coordinate system.

B.6 POWERS AND ROOTS

The complex number raised to the nth power is defined by

$$z^n = |z|^n\, e^{jn\theta} \tag{20}$$

Thus, again we note the usefulness of the polar coordinate representation.

It is also interesting to note that

$$e^{j\theta} \equiv e^{j(\theta+2\pi k)} = e^{j\theta}\, e^{j2\pi k}, \quad k \text{ an integer} \tag{21}$$

since $\cos(2\pi k) = 1$ and $\sin(2\pi k) = 0$. With this information we can now define the nth root of z. For example, consider the equation

$$z^3 + 1 = 0 \tag{22}$$

We can write

$$z^3 = -1 = e^{j180°} \tag{23}$$

therefore

$$z = (e^{j180°})^{1/3} = e^{j60°} = \frac{1}{2} + j\frac{\sqrt{3}}{2} \tag{24}$$

Note that $z = -1$ is also a solution to (22). Hence, Equation 23 does not locate all of the roots. Therefore, let us use identity (21). Then

$$z = (e^{j(180°+2\pi k)})^{1/3}, \quad k \text{ an integer}$$

This is the same as

$$z = e^{j(60°+120°k)} \tag{25}$$

For $k = 0$ we obtain

$$z_1 = e^{j60°} = \frac{1}{2}(1 + j\sqrt{3}) \tag{26}$$

For $k = 1$,

$$z_2 = e^{j180°} = -1 \tag{27}$$

and for $k = 2$

$$z_3 = e^{j300°} = \frac{1}{2}(1 - j\sqrt{3}) \tag{28}$$

These three roots lie on the unit circle in Figure B.3. Note that the third root is the complex conjugate of the first. In any polynomial with real coefficients the complex roots must appear in conjugate pairs. Convince yourselves that any other integer values of k result in the same roots.

Thus we define the nth root of a complex number as

$$z^{1/n} = |z|^{1/n} e^{j(\theta+2\pi k/n)}, \quad k = 0, 1, 2, \ldots, n-1 \tag{29}$$

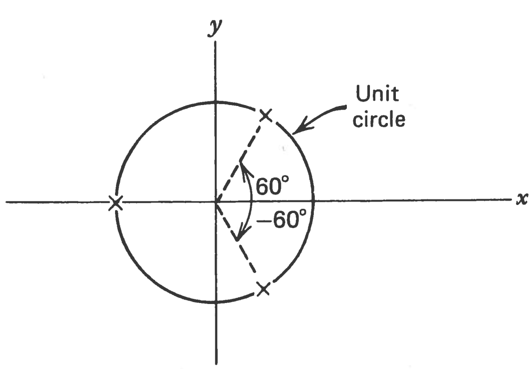


Figure B.3. Roots of $z^3 + 1 = 0$ in the complex plane.

PROBLEMS

1. Show that $zz^* = |z|^2$.
2. Prove the following identities:

   (a) $\text{Re}[e^{x+jy}] = e^x \cos y$
   (b) $\text{Im}[e^{x+jy}] = e^x \sin y$
   (c) $e^{jx} + e^{-jx} = 2 \cos x$
   (d) $e^{jx} - e^{-jx} = 2j \sin x$
3. Prove that $(\cos \theta + j \sin \theta)^n = \cos n\theta + j \sin n\theta$.
4. Convert to polar form

   (a) $1 + j2$ (b) $-5 + j3$
   (c) $-3 - j5$ (d) $2 - j1$
   (e) $0.1 + j10$ (f) $5.6 - j0.1$
5. Convert to rectangular form
   (a) $10\underline{/30^\circ}$ (b) $5\underline{/205^\circ}$
   (c) $1\underline{/60^\circ}$ (d) $8.2\underline{/-45^\circ}$
6. Evaluate the following expression and give your answer in rectangular form.

   (a) $z = \dfrac{10\underline{/70^\circ} - 5\underline{/10^\circ}}{1 + j2}$

   (b) $z = \ln(5\underline{/90^\circ})$

   (c) $z = \dfrac{(1 - j2)(-3 + j1)}{3 + j4}$

   (d) $z = 4\, e^{1+j}$
7. Find the roots of the following equations:

   (a) $z^2 + 1 = 0$ (b) $z^2 - 1 = 0$
   (c) $z^3 - 1 = 0$ (d) $z^5 + 1 = 0$
8. Solve the following equations for the unknown constants

   (a) $(10 + j20)2\underline{/\theta} = K\underline{/30^\circ}$
   (b) $2\underline{/-120^\circ} + 3.5 + jb = a + j5$
9. Show that Equations 15 and 16 are equivalent.
10. Show that Equations 17 and 19 are equivalent.
11. Show that

    $$\text{Re}[z] = \frac{1}{2}(z + z^*)$$

    $$\text{Im}[z] = \frac{1}{2j}(z - z^*)$$
12. Any polynomial with real coefficients a_i can be factored as

    $$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_o = (x - z_1)(x - z_2) \cdots (x - z_n)$$

    where the z_i's are the roots or zeros of the polynomial. Show that if z_i is a complex root (imaginary part not zero) then there must be another root $z_j = z_i^*$ for some $j \neq i$, that is, the complex roots of a polynomial with all real coefficients must occur in conjugate pairs.