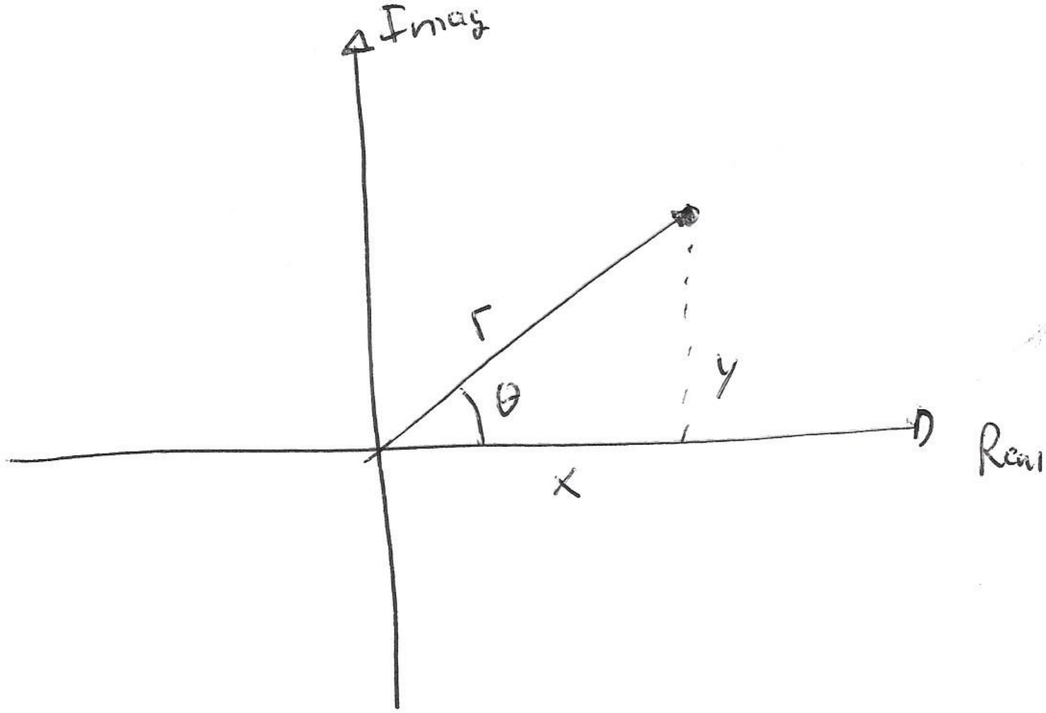
Complex Plane

$Z = x + jy$



$r^2 = x^2 + y^2$

$\text{atan}(y/x) = \theta$

$Z = r\cos\theta + r\sin\theta\, j$

Can give Complex # Z as $\quad Z = r\angle\theta$

Taylor Series

$$f(x) = f(0) + \frac{df}{dx}\Big|_{x=0} x + \frac{d^2f}{dx^2}\Big|_{x=0} \frac{x^2}{2} + \frac{d^3f}{dx^3}\Big|_{x=0} \frac{x^3}{3!} + \ldots \frac{d^nf}{dx^n}\Big|_{x=0} \frac{x^n}{n!}$$

Try for $f(x) = e^x$

where $\frac{d}{dx}e^x = e^x$; $\frac{d^2}{dx^2}e^x = e^x$

$$e^x = e^0 + e^0 x + e^0 \frac{x^2}{2} + e^0 \frac{x^3}{3!} + \ldots e^0 \frac{x^n}{n!}$$

$$\boxed{e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots \frac{x^n}{n!}}$$

Try $\cos(x)$

where $\frac{d}{dx}\cos(x) = -\sin x$ $\quad \frac{d^2}{dx^2} = -\cos(x)$ $\quad \frac{d^3}{dx^3} = \sin(x)$ . . .

$$\cos x = \cos(0) - \sin(0)x - \cos(0)\frac{x^2}{2} + \sin(0)\frac{x^3}{3!} + \cos(0)\frac{x^4}{4!} + \ldots$$

$$= 1 - 0 - \frac{x^2}{2} + 0 + \frac{x^4}{4!}$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots$$

Try Sin(x)

Where $\frac{d\,\mathrm{Sin}(x)}{dx} = \mathrm{Cos}\,x \qquad \frac{d^2}{dx^2} = -\mathrm{Sin}(x)$

$$\sin x = \mathrm{Sin}(0) + \mathrm{Cos}(0)\,x - \mathrm{Sin}(0)\frac{x^2}{2} - \mathrm{Cos}(0)\frac{x^3}{3!}$$

$$= 0 + x - 0 - \frac{x^3}{3!} + \ldots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

Try e^{jx}

$\frac{d}{dx}e^{jx} = je^{xj} \qquad \frac{d^2}{dx^2} = -e^{jx} \qquad \frac{d^3}{dx^3} = -je^{jx} \qquad \frac{d^4}{dx^4} = e^{jx}$

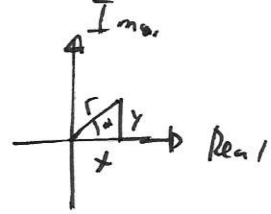
$$e^{jx} = e^{j\cdot 0} + je^{j0}x - e^{j0}\frac{x^2}{2} - je^{j0}\frac{x^3}{3!} + e^{j0}\frac{x^4}{4!} + je^{j0}\frac{x^5}{5!} - e^{j0}\frac{x^6}{6!} \ldots$$

$$e^{jx} = 1 + jx - \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} + j\frac{x^5}{5!} - \frac{x^6}{6!}$$

$$e^{jx} = \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) + j\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)$$

$$\boxed{e^{jx} = \cos x + j\sin x}$$

Therefore



$$Z = x + jy$$

$$Z = r\cos\theta + j\sin\theta$$

$$Z = r(\cos\theta + j\sin\theta) = re^{j\theta}$$

Complex multiplication

$$Z_1 \cdot Z_2 = r_1 e^{j\theta_1} r_2 e^{j\theta_2}$$

$$= r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

in polar form, multiplication is easy, angles add

Division

$$\frac{Z_1}{Z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$$

angles subtract!

Let's assume a function

$$V(t) = (x + jy)\, e^{j\omega t}$$

$$V(t) = r e^{j\theta} e^{j\omega t}$$ where $r = \sqrt{x^2 + y^2}$

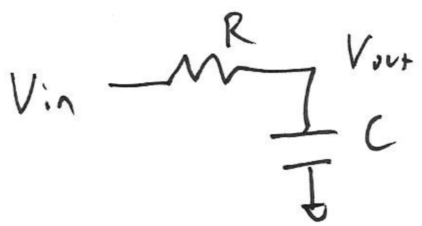
$\theta = \operatorname{atan}(y/x)$

$$V(t) = r e^{j(\omega t + \theta)}$$

$$V(t) = r\cos(\omega t + \theta) + r\sin(\omega t + \theta)\, j$$

If we assume Voltage has form $e^{j\omega t}$
then we have assumed Sines and Cosines
where the complex constant out front contains
the Amplitude of the signal and the angle.

Example



$$i = \frac{V_{in} - V_o}{R} = C\frac{dV_o}{dt}$$

$$RC\frac{dV_o}{dt} = V_{in} - V_{out}$$

Let's Assume $V_{in}(t) = 1 \cdot e^{j\omega t} = \cos\omega t + j\sin\omega t$

$$V_{out}(t) = \bar{V}_o e^{j\omega t} = |\bar{V}_o|\cos(\omega t+\theta) + j\sin(\omega t+\theta)$$

$\uparrow$ Complex #

$$-RC\,|\bar{V}_o|\,\omega\sin(\omega t+\theta) + RC\,|\bar{V}_o|\,\omega\cos(\omega t+\theta)\,j =$$

$$\cos(\omega t) + j\sin(\omega t) - |\bar{V}_o|\cos(\omega t+\theta) - j\sin(\omega t+\theta)$$

real + Imag must be zero

$$-RC\omega|\bar{V}_o|\sin(\omega t+\theta) = \cos(\omega t) - |\bar{V}_o|\cos(\omega t+\theta)$$

$$RC\omega|\bar{V}_o|\cos(\omega t+\theta) = \sin(\omega t) - |\bar{V}_o|\sin(\omega t+\theta)$$

Previously Solved "Red" problem

Solution was

$$\tan(\theta) = -RC\omega$$

$$|V_o| = \frac{1}{\cos\theta - RC\omega \sin\theta}$$

Expand "Blue" problem w/ trig.

$$-RC\omega|V_o|\left(\sin(\omega t)\cos\theta + \cos\omega t \sin\theta\right) = \cos\omega t -$$

$$|V_o|\left(\cos\omega t \cos\theta - \sin\omega t \sin\theta\right)$$

group $\sin \omega t$

$$-RC\omega \cancel{|V_o|} \cancel{\sin}(\omega t) \cos\theta = + \cancel{|V_o|} \cancel{\sin\omega t} \sin\theta$$

$$\tan\theta = -RC\omega$$ Same as "Red" problem.

group $\cos\omega t$

$$-RC\omega|V_o| \cancel{\cos\omega t} \sin\theta = \cancel{\cos(\omega t)} - |V_o| \cancel{\cos\omega t} \cos\theta$$

$$|V_o| = \frac{1}{\cos\theta - RC\omega \sin\theta}$$

So

Solution for $\sin(\omega t)$ input is the same as for $\cos(\omega t)$!

$V_{in}(t) = \cos \omega t + j \sin \omega t$

works because we solve real and imag. parts to get a single answer.

Now, Let's use Complex #s

$V_{in} = e^{j\omega t}$

$V_{out}(t) = \bar{V}_o e^{j\omega t}$

Complex #

$RC \frac{d}{dt} V_o(t) = V_{in}(t) - V_o(t)$

$RCj\omega \bar{V}_o e^{j\omega t} = e^{j\omega t} - \bar{V}_o e^{j\omega t}$

$$\boxed{\bar{V}_o = \frac{1}{1 + RCj\omega}}$$

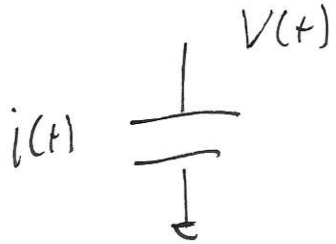
Much easier than using trig.

recall Division is easiest in polar form

$$V_o = \frac{1}{1 + RC\omega j} = \frac{1}{\sqrt{1+R^2C^2\omega^2}\, e^{j\,\text{atan}(RC\omega)}}$$

$$V_o = \underbrace{\frac{1}{\sqrt{1+R^2C^2\omega^2}}}_{\text{Amplitude part of Bode plot}} e^{-j\,\underbrace{\text{atan}(RC\omega)}_{\text{angle, or phase of the Bode Plot.}}}$$

Impedance



Complex #

$$V(t) = V e^{j\omega t}$$

$$i(t) = I e^{j\omega t}$$

Complex #

Capacitor:

$$i(t) = C \frac{dV(t)}{dt}$$

$$I e^{j\omega t} = C \frac{d}{dt} V e^{j\omega t} = \underbrace{\boxed{j\omega C V}}_{I} e^{j\omega t}$$

Define

$$\frac{V(t)}{i(t)} = \text{impedance}, Z$$

$$\frac{V(t)}{i(t)} = \frac{V e^{j\omega t}}{I e^{j\omega t}} = \frac{V}{I} = \frac{V}{j\omega C V} = \boxed{\frac{1}{j\omega C}}$$

Impedance of Capacitor.

Impedance analysis

To Solve circuits w/ capacitors, it is just like Solving problems w/ resistors.

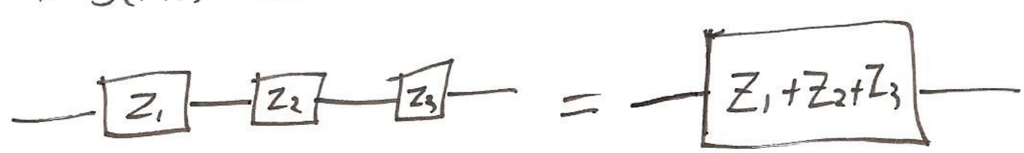
Now we use

$$V = I Z$$

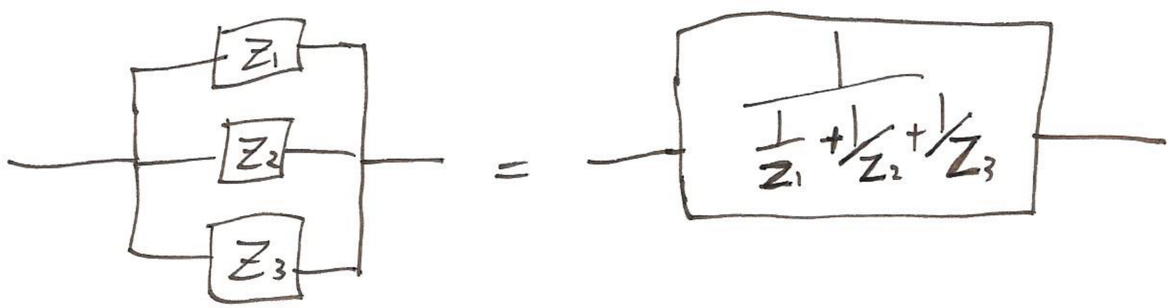
where Z for Resistor $Z = R$

Capacitor $Z = \frac{1}{j\omega C}$

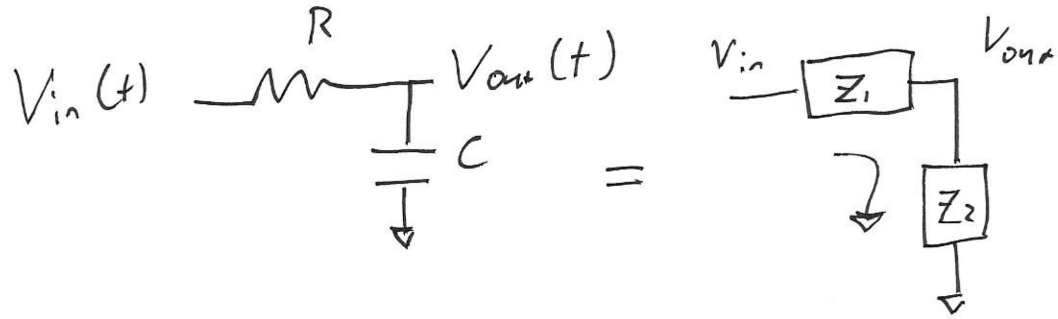
impedance in Series add:



impedance in parallel Combine too,



Example



Voltage Divider

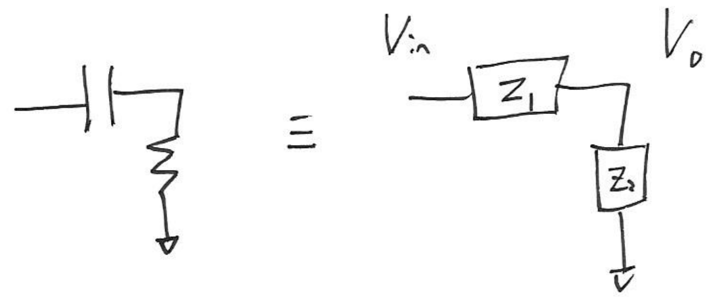
$$i = \frac{\tilde{V}_{in} - \tilde{V}_o}{Z_1} = \boxed{\frac{\tilde{V}_{in} - 0}{Z_1 + Z_2} = \frac{\tilde{V}_{out}}{Z_2}}$$

$$\frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{Z_2}{Z_1 + Z_2} = \frac{1/j\omega C}{R + 1/j\omega C}$$

$$\boxed{\frac{\tilde{V}_o}{\tilde{V}_{in}} = \frac{1}{1 + RC\omega j}}$$

Same result as before!

Let's Switch the parts



Same as last circuit

$$\frac{V_o}{V_{in}} = \frac{Z_2}{Z_1 + \cancel{P} Z_2} = \frac{R}{R + \frac{1}{j\omega C}}$$

$$\boxed{\frac{V_o}{V_{in}} = \frac{RC\omega j}{1 + RC\omega j}}$$

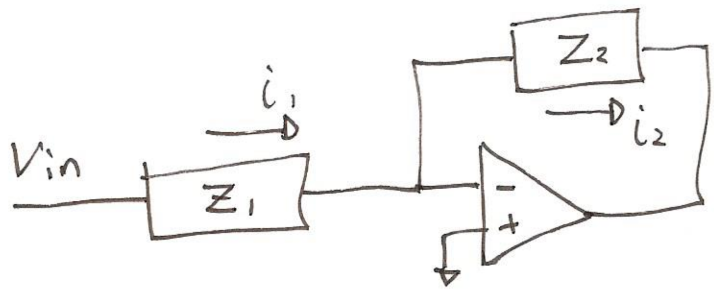
High pass filter $RC\omega \to \infty$

$$\frac{V_o}{V_{in}} \Rightarrow 1$$

$$RC\omega \to 0$$

$$\frac{V_o}{V_{in}} \approx RC\omega j$$

Example w/ op-amp



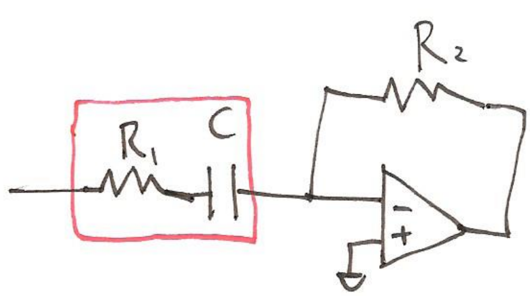
rules for Op-amp

- $i_+ = i_- = 0$ thus $i_1 = i_2$
- $V_- = V_+ = 0$.

Thus

$$i = \frac{V_{in}}{Z_1} = \frac{-V_{out}}{Z_2} \qquad \frac{V_{out}}{V_{in}} = \frac{-Z_2}{Z_1}$$

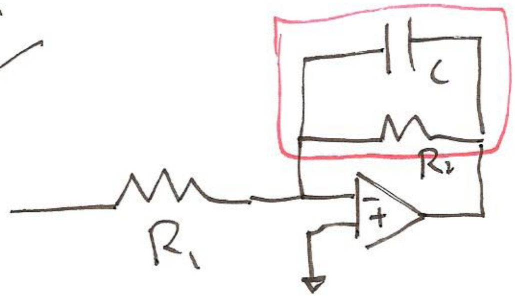
So/



$$Z_1 = R_1 + \frac{1}{j\omega C}$$
$$Z_2 = R_2$$

$$\frac{V_{out}}{V_{in}} = \frac{-R_2}{R_1 + \frac{1}{j\omega C}} = \underbrace{\frac{-R_2}{R_1}}_{\text{Amplify}} \underbrace{\left(\frac{j\omega R_1 C}{1 + j\omega R_1 C}\right)}_{\text{High pass}}$$

Or

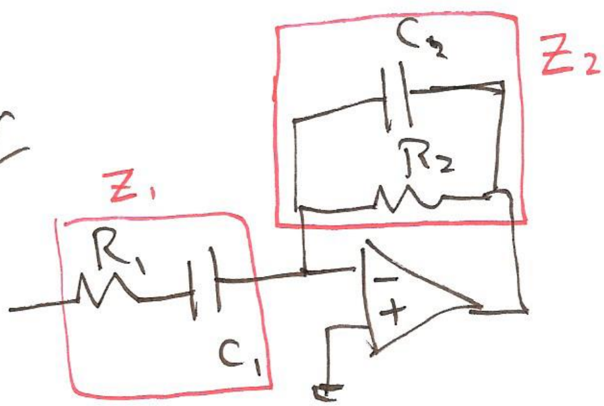


$$Z_2 = \frac{1}{\frac{1}{R_2} + \frac{1}{1/j\omega C}}$$

$$Z_2 = \frac{R_2}{1 + j\omega R_2 C}$$

So $$\frac{V_{out}}{V_{in}} = \frac{-Z_2}{Z_1} = \underbrace{-\frac{R_2}{R_1}}_{\text{Amplifier}} \underbrace{\frac{1}{1 + j\omega R_2 C}}_{\text{Low Pass}}$$

Or



Note Z_2, Z_1 found in prior problem!

$$\frac{V_{out}}{V_{in}} = \frac{-Z_2}{Z_1} = \frac{-R_2}{1 + j\omega R_2 C} \cdot \frac{1}{R_1 + 1/j\omega C_1}$$

$$= -\left(\frac{R_2}{R_1}\right)\left(\frac{1}{1 + j\omega R_2 C}\right)\left(\frac{j\omega R_1 C_1}{1 + j\omega R_1 C_1}\right)$$

Amp   Low Pass   High Pass