Euler Identity

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Proof

Taylor series for e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \dots$$

So

$$e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - j\frac{\theta^7}{7!}$$

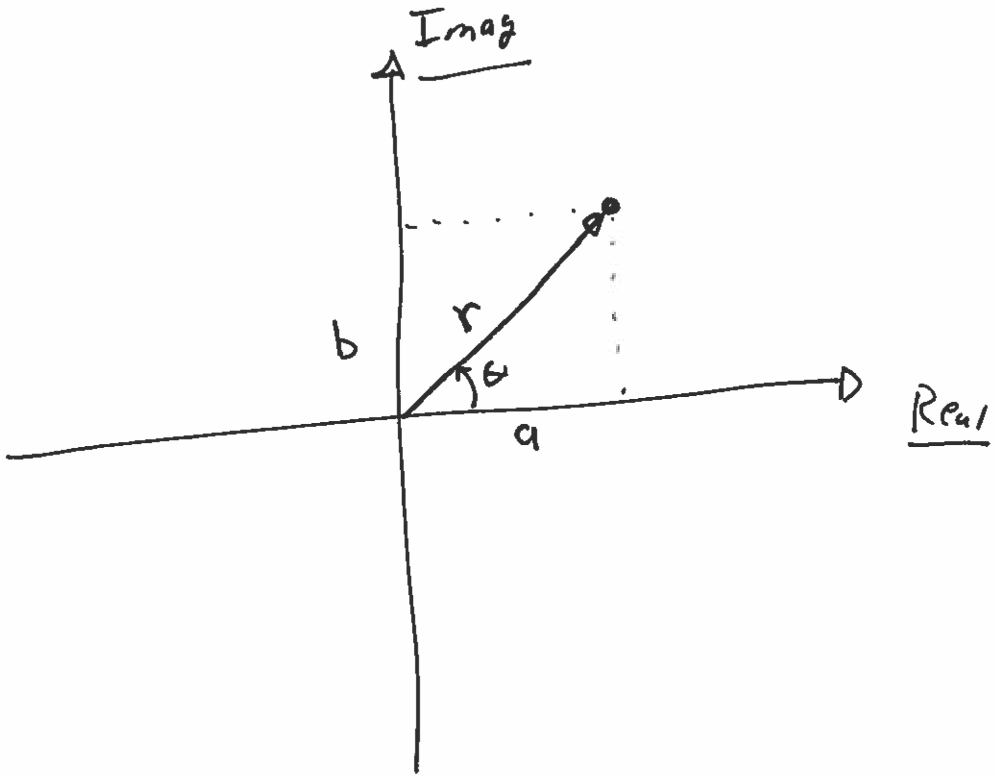
$$e^{j\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right) + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right)$$

$\Downarrow$ Taylor Series

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Taylor Series

Plots in Complex plane



Complex # is: $a + bj$

$$r = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$a = r\cos\theta$$

$$b = r\sin\theta$$

So

$$a + bj = r\cos\theta + j\,r\sin\theta = re^{j\theta}$$

Let's assume we can represent a time varying, sinusoidal voltage as

$$\boxed{V(t) = \tilde{V} e^{j\omega t}}$$

where $\tilde{V} = a + bj$ is a complex amplitude

ω is the frequency, $r = \sqrt{a^2+b^2}$

Connection to Sines/Cosines

$$V(t) = (a+bj)(\cos\omega t + j\sin\omega t)$$

$$V(t) = (a\cos\omega t - b\sin\omega t) + j(b\cos\omega t + a\sin\omega t)$$

$$V(t) = r\left(\frac{a}{r}\cos\omega t - \frac{b}{r}\sin\omega t\right) + jr\left(\frac{b}{r}\cos\omega t + a/r\sin\omega t\right)$$

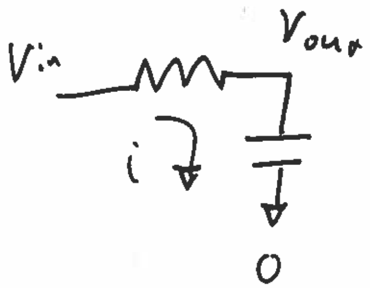
$$V(t) = r(\cos\theta\cos\omega t - \sin\theta\sin\omega t) + jr(\sin\theta\cos\omega t + \cos\theta\sin\omega t)$$

$\Downarrow$ trig Identities

$$\boxed{V(t) = r\cos(\omega t + \theta) + jr\sin(\omega t + \theta)}$$

So putting $V(t) = \tilde{V} e^{j\omega t}$ is equivalent to using sines + cosines. The magnitude of $|\tilde{V}| = \sqrt{a^2+b^2} = r$ is the amplitude.

Example



$$i = \frac{V_{in} - V_{out}}{R} = C \frac{dV_{out}}{dt}$$

$$RC \frac{dV_{out}}{dt} = V_{in} - V_{out}$$

<u>assume</u>

$$V_0(t) = \bar{V}_0 e^{j\omega t} = |\bar{V}_0| \left(\cos(\omega t + \theta) + j \sin(\omega t + \theta) \right)$$

$$V_{in}(t) = \bar{V}_{in} e^{j\omega t} = |\bar{V}_{in}| \left(\cos(\omega t + \theta) + j \sin(\omega t + \theta) \right)$$

$$RCj\omega \bar{V}_0 e^{j\omega t} = \bar{V}_{in} e^{j\omega t} - \bar{V}_0 e^{j\omega t}$$

$e^{j\omega t}$ cancels

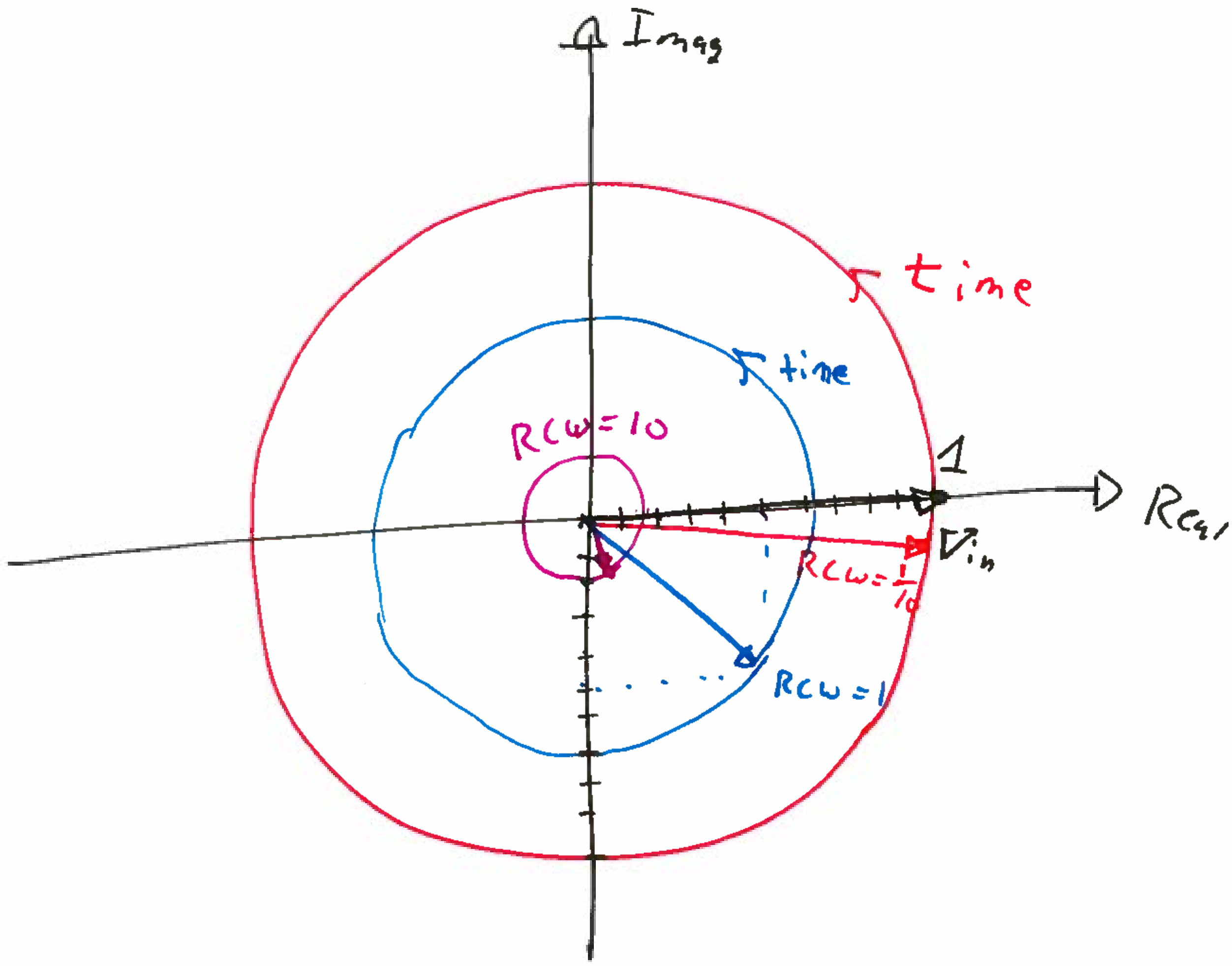
$$(RCj\omega + 1) \bar{V}_0 = \bar{V}_{in}$$

$$\frac{\bar{V}_0}{\bar{V}_{in}} = \frac{1}{1 + RC\omega j} = \frac{1}{1 + RC\omega j} \cdot \frac{1 - RC\omega j}{1 - RC\omega j}$$

$$\boxed{\frac{\bar{V}_0}{\bar{V}_{in}} = \frac{1 - RC\omega j}{1 + (RC\omega)^2}}$$

Let's assum

$\tilde{V}_{in} = 1$



Cases

$RC\omega = \frac{1}{10} \Rightarrow \tilde{V}_{out} = \dfrac{1 - \frac{j}{10}}{1.01} = \dfrac{1}{1.01} - \dfrac{j}{(10)\cdot 1.01}$

$RC\omega = 1 \Rightarrow \tilde{V}_{out} = \dfrac{1-j}{2} \qquad |V_o| = \frac{1}{\sqrt{2}}$

$RC\omega = 10 \Rightarrow \tilde{V}_{out} = \dfrac{1-10j}{1+10^2} = \dfrac{1}{101} - \dfrac{10j}{101}$

Analytically

$$V_o : \frac{1 - RC\omega j}{(1 + (RC\omega)^2)}$$

$$|V_o| = \sqrt{\frac{1}{(1 + (RC\omega)^2)^2} + \frac{(RC\omega)^2}{(1 + (RC\omega)^2)^2}}$$

$$= \sqrt{\frac{1 + (RC\omega)^2}{(1 + (RC\omega)^2)^2}}$$

Derived in earlier course notes

$$|V_o| = \sqrt{\frac{1}{1 + (RC\omega)^2}}$$

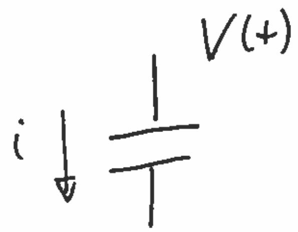
Imag Part!

$$\Theta = \operatorname{atan}\left(\frac{\frac{-RC\omega}{1 + (RC\omega)^2}}{\frac{1}{1 + (RC\omega)^2}}\right) = \operatorname{atan}(-RC\omega)$$

Real Part!

Same result as before

Impedance



$$V(t) = \bar{V} e^{j\omega t}$$

$$i = C \frac{dV}{dt}$$

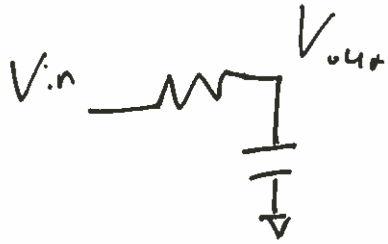
$$i = C\bar{V} j\omega e^{j\omega t}$$

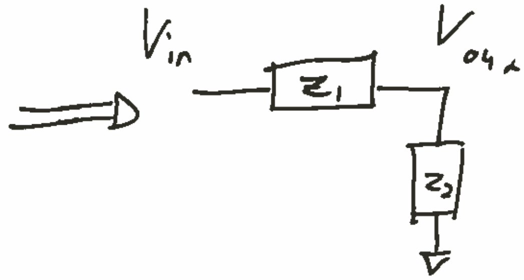
So

$$\frac{\bar{V}(t)}{i(t)} = \frac{\bar{V} e^{j\omega t}}{C\bar{V} j\omega e^{j\omega t}} = \frac{1}{j\omega C} = Z \text{ (impedance)}$$

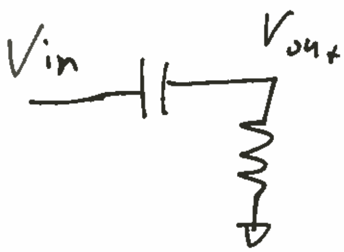
- By definition we take the ratio of Voltage and current to be the impedance
- It turns out using impedances we can analyze circuits just like it was resistors.

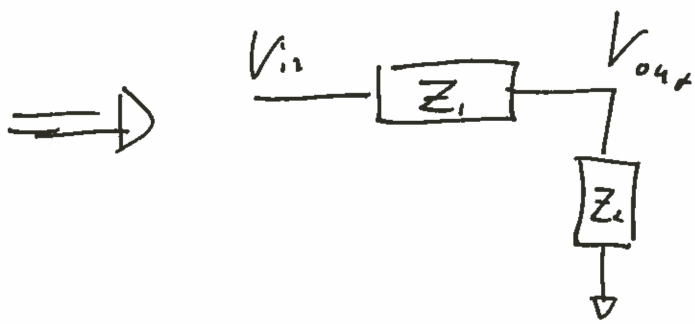
Examples



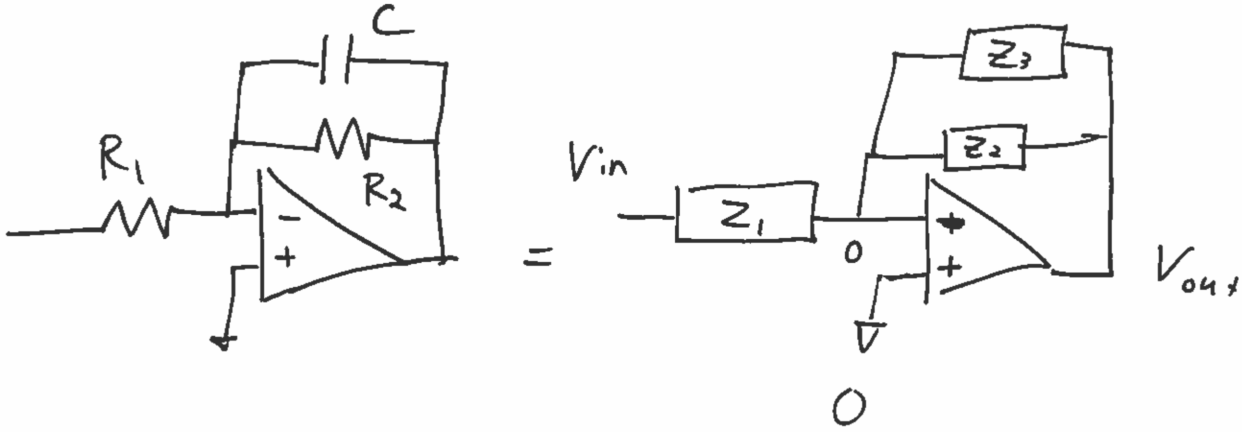


$$\frac{V_{out}}{V_{in}} = \frac{Z_2}{Z_1 + Z_2} = \frac{1/j\omega C}{R + 1/j\omega C} = \boxed{\frac{1}{1 + j\omega RC}}$$





$$\frac{V_{out}}{V_{in}} = \frac{Z_2}{Z_1 + Z_2} = \frac{R}{R + 1/j\omega C} = \boxed{\frac{j\omega RC}{1 + j\omega RC}}$$



Same analysis as with resistors.

$$i_1 = \frac{V_{in}}{Z_1}, \quad i_2 = \frac{-V_{out}}{Z_2} \qquad i_3 = \frac{-V_{out}}{Z_3}$$

$$i_1 = i_2 + i_3$$

$$\frac{V_{in}}{Z_1} = -V_{out}\left(\frac{1}{Z_2} + \frac{1}{Z_3}\right)$$

$$\frac{V_{out}}{V_{in}} = -\frac{1/Z_1}{1/Z_2 + 1/Z_3} = \frac{-1/R_1}{1/R_2 + j\omega C}$$

$$\boxed{\frac{V_{out}}{V_{in}} = \frac{-R_2/R_1}{1 + j\omega C R_2}}$$