Hydraulic analogy for resistors and capacitors

Physical system

Imagine a tank of water connected by a pipe at the bottom open to the atmosphere. The tank has a cross sectional area, A, and height H. The pipe has diameter D and length L. Water flows from the left to the right until the tank is empty. In this system the pressure at the bottom of the tank is given by $P = \rho g H$ where ρ is the density of water (1000 kg/m^3), g is 9.8 m/s^2, and H is the height. Since the pipe drains the tank to the atmosphere, the pressure at the pipe's exit is $P = 0$.

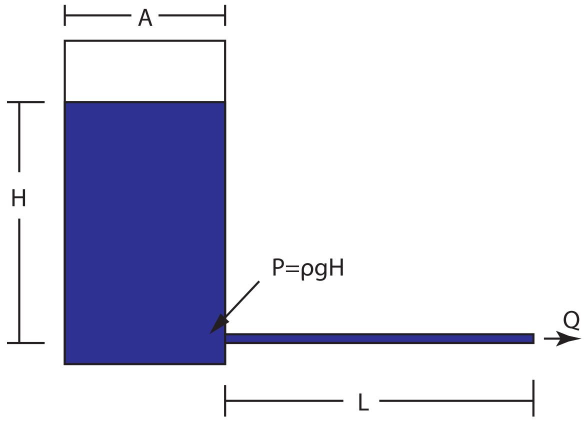

FIG. 1 Schematic of a simple hydraulic system.

Resistance

The volumetric flow rate of water, Q, through a pipe is related to the pressure drop applied *across* the pipe, ΔP. The overall pressure is unimportant - it is the difference in pressure from inlet to outlet. In this simple example, the applied pressure drop is $\Delta P = \rho g H$. The higher the applied pressure, the higher the flow rate. It seems obvious that the flow rate also depends upon the size and length the pipe. A long skinny tube would have lower flow than a short and fat one. Imagine a case where the cross sectional area of the tank is large compared to the volumetric flow rate of water. In this case, the height of the tank changes very slowly in time. We can imagine with this arrangement we could set up and experiment where we adjust the water tank's height and measure the resulting flow rate. To measure the flow rate, we could simply collect water in a measuring cup for a fixed amount of time. If we conducted this experiment we could make a plot of pressure versus flow.

We would normally describe the pipe's resistance, R, as the ratio of pressure to flow;

$$R(Q) = \frac{\Delta P}{Q}.$$

This definition simply says that if the pressure is high and the flow is low, the resistance is high. In general, the resistance itself can depend upon the flow rate denoted by $R(Q)$, resistance as a function of flow rate. If our measured plot of pressure versus flow turned out to be linear, then we have a simple relationship where R is constant,

$$\Delta P = RQ.$$

In the hydraulic case, it turns out in some cases the resistance is constant and in others it is not. For our purposes, we will assume we measured a constant resistance to flow.

Capacitance

Now let's go to the case where we fill the tank and monitor the water height as a function of time as the tank drains. As the water flows, the tank empties and the pressure applied to the pipe decreases. As the pressure decreases, so does the flow. Since the rate that the water level changes is proportional to the flow rate, the rate of height change slows. The result is that the water level approaches zero as shown in Figure 2.

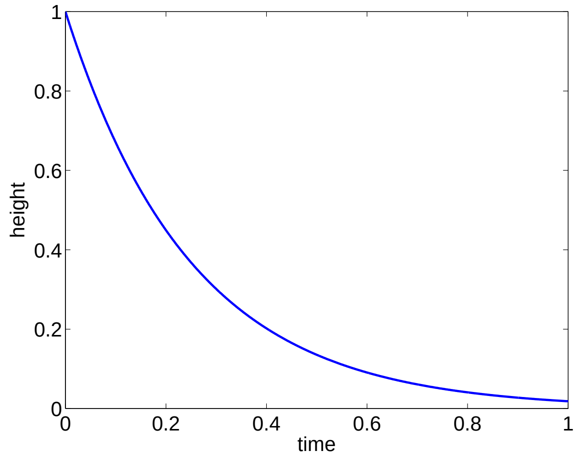

FIG. 2 Behavior of tank draining problem. As the height decreases, so does the pressure and thus the flow.

To be a little more precise, the rate of change of the volume of fluid in the tank is equal to the flow rate through the tube,

$$\frac{dV}{dt} = A\frac{dH}{dt} = -Q.$$

The negative sign denotes that the height decreases when the flow is from left to right. The pressure drop across the pipe is related to the height of water in the tank,

$$\Delta P = \rho g H.$$

We also have that for a constant resistance that the flow and pressure are related as

$$\Delta P = QR.$$

Therefore the flow rate is related to the height as

$$Q = \frac{\rho g H}{R}.$$

Substituting this result into our rate equation yields,

$$\frac{dH}{dt} = -\frac{\rho g}{AR}H = -\frac{H}{\tau}.$$

Note that the parameter, $\frac{AR}{\rho g}$, must have units of time. We call this the time constant for the system, $\tau = \frac{AR}{\rho g}$.

In order to solve this simple differential equation, we separate the variables,

$$\frac{dH}{H} = -\frac{dt}{\tau},$$

and integrate

$$\ln(H) = -\frac{t}{\tau} + C.$$

where C is a constant of integration. Taking the exponential of both sides,

$$H = \mathrm{e}^{-t/\tau + C} = \mathrm{e}^{C}\mathrm{e}^{-t/\tau} = B\mathrm{e}^{-t/\tau},$$

where B is just some other constant. We obtain the constant of integration by using the initial state of the system, namely $H(t = 0) = H_0$, the initial height. Thus

$$H = H_0\mathrm{e}^{-t/\tau}.$$

The height falls exponentially and the time scale on which it does so is τ. Note that the time constant increases with resistance and area of the tank. The area of the tank is related to the tank's size or *capacity*.

The time constant determines the characteristic time it takes for the tank to drain. In Figure 3 we show a time constant of $\tau = 1$ and $\tau = 3$. If one draws a straight line from the initial height at $t = 0$ to the value of τ on the time axis, this straight line will have the same slope as the exponential at $t = 0$.

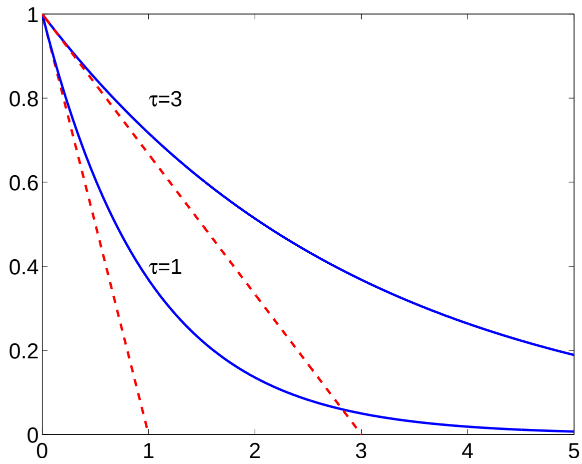

FIG. 3 Effect of an increasing time constant on the resulting dynamics. Here we show results for $\tau = 1$ and $\tau = 3$.